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Yangians and quantum loop algebras

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Abstract Let $\mathfrak{g}$ be a complex, semisimple Lie algebra. Drinfeld showed that the quantum loop algebra $U_\hbar(L\mathfrak{g})$ of $\mathfrak{g}$ degenerates to the Yangian $Y_\hbar(\mathfrak{g})$. We strengthen this result by constructing an explicit algebra homomorphism Φ from $U_\hbar(L\mathfrak{g})$ to the completion of $Y_\hbar(\mathfrak{g})$ with respect to its grading. We show moreover that Φ becomes an isomorphism when $U_\hbar(L\mathfrak{g})$ is completed with respect to its evaluation ideal. We construct a similar homomorphism for $\mathfrak{g} = \mathfrak{gl}_n$ and show that it intertwines the actions of $U_\hbar(L\mathfrak{gl}_n)$ and $Y_\hbar(\mathfrak{gl}_n)$ on the equivariant K-theory and cohomology of the variety of n-step flags in $\mathbb{C}^d$ constructed by Ginzburg–Vasserot.

Keywords Affine quantum groups · Yangian · Quantum loop algebra

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Contents

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1 Introduction

1.1

The present paper is motivated by and lays the groundwork for a proof of the trigonometric monodromy conjecture formulated by the second author in [32]. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, G the corresponding connected and simply-connected Lie group, $H \subset G$ a maximal torus and W the corresponding Weyl group. In [32] a flat, W-equivariant connection $\widehat{\nabla}_C$ was constructed on H which has logarithmic singularities on the root subtori of H and values in any finite-dimensional representation of the Yangian $Y_\hbar(\mathfrak{g})$. By analogy with the description of the monodromy of the rational Casimir connection of $\mathfrak{g}$ obtained in [30,31], it was conjectured in [32] that the monodromy of the trigonometric Casimir connection $\widehat{\nabla}_C$ is described by the action of the affine braid group of $\mathfrak{g}$ arising from the quantum Weyl group operators of the quantum loop algebra $U_\hbar(L\mathfrak{g})$. This raises in particular the problem of relating finite-dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_\hbar(L\mathfrak{g})$.

1.2

Since their construction by Drinfeld [9,10], these affine quantum groups have been extensively studied from several perspectives (see, e.g., [5, chap. 12], [2] and references therein) and are widely believed to share the same finite-dimensional representation theory. This belief is corroborated in part by the following facts

(1) The quantum loop algebra $U_\hbar(L\mathfrak{g})$ degenerates to the Yangian $Y_\hbar(\mathfrak{g})$. Specifically, if $U_\hbar(L\mathfrak{g})$ is filtered by the powers of the evaluation ideal at $z=1$, its associated graded is isomorphic to $Y_\hbar(\mathfrak{g})$ [10,17].
(2) Finite-dimensional simple modules over $U_\hbar(L\mathfrak{g})$ are parametrised by $\mathbf{I}$-tuples of (Drinfeld) polynomials $\{P_i(u)\}_{i\in\mathbf{I}}$ satisfying $P_i(0) = 1$, where $\mathbf{I}$ is the set of vertices of the Dynkin diagram of $\mathfrak{g}$ [4]. Similarly, finite-dimensional simple modules over $Y_\hbar(\mathfrak{g})$ are classified by $\mathbf{I}$-tuples of monic polynomials [3,11,28,29].
(3) If $\mathfrak{g}$ is simply laced, there exists, for every $\mathbf{w} \in \mathbb{N}^{\mathbf{I}}$, a Steinberg variety $Z(\mathbf{w})$ endowed with an action of $GL(\mathbf{w}) \times \mathbb{C}^\times$ (here $GL(\mathbf{w}) = \prod_{i\in\mathbf{I}} GL_{w_i}$), and algebra homomorphisms

$$\Psi_U : U_\hbar(L\mathfrak{g}) \to K^{GL(\mathbf{w})\times\mathbb{C}^\times}(Z(\mathbf{w}))$$
$$\Psi_Y : Y_\hbar(\mathfrak{g}) \to H^{GL(\mathbf{w})\times\mathbb{C}^\times}(Z(\mathbf{w}))$$

The variety $Z(\mathbf{w})$ and the homomorphism Ψ_U were constructed by Nakajima [26], while Ψ_Y was constructed by Varagnolo [33].

1.3

The above results go some way towards relating the categories of finite-dimensional representations of $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$. For example, exponentiating the roots of Drinfeld polynomials yields, via (2), a surjective map $\exp^*$ between the set of isomorphism classes of irreducible finite-dimensional modules of $Y_\hbar(\mathfrak{g})$ and those of $U_\hbar(L\mathfrak{g})$-modules. If $\mathfrak{g}$ is simply laced, the geometric realisations (3) imply further that $\exp^*$ preserves the dimensions of these representations [33].

Despite these results however, and to the best of our knowledge, no natural relation between the categories of finite-dimensional representations of $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ is known. Part of the difficulty in exploiting, say, the geometric realisations to pursue this question lies in the fact that the homomorphisms Ψ_U, Ψ_Y are neither injective nor surjective [26]. Moreover, although these realisations yield all irreducible representations, the categories $\mathrm{Rep}_{\mathrm{fd}}(U_\hbar(L\mathfrak{g}))$ and $\mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ are not semisimple.

1.4

The aim of the present paper is to clarify the relation between $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$. We do so by constructing an explicit algebra homomorphism

$$\Phi : U_\hbar(L\mathfrak{g}) \longrightarrow \widehat{Y_\hbar(\mathfrak{g})}$$

where $\widehat{Y_\hbar(\mathfrak{g})}$ is the completion of $Y_\hbar(\mathfrak{g})$ with respect to its $\mathbb{N}$-grading and show that it induces an isomorphism of completed algebras. We also show that Φ exponentiates the roots of Drinfeld polynomials, though we defer the study of the corresponding pull-back functor

$$F = \Phi^* : \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \to \mathrm{Rep}_{\mathrm{fd}}(U_\hbar(L\mathfrak{g}))$$

to the sequel of this paper [13].

To state out results more precisely, recall that $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ are deformations of the loop and current algebras $U(\mathfrak{g}[z,z^{-1}])$ and $U(\mathfrak{g}[s])$ respectively. Denote by

$$U_\hbar(L\mathfrak{h}), U_\hbar(L\mathfrak{b}_\pm) \subset U_\hbar(L\mathfrak{g}) \qquad \text{and} \qquad Y_\hbar(\mathfrak{h}), Y_\hbar(\mathfrak{b}_\pm) \subset Y_\hbar(\mathfrak{g})$$

the subalgebras deforming $U(\mathfrak{h}[z,z^{-1}])$, $U(\mathfrak{b}_\pm[z,z^{-1}])$ and $U(\mathfrak{h}[s])$, $U(\mathfrak{b}_\pm[s])$ respectively, where $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of H and $\mathfrak{b}_\pm \subset \mathfrak{g}$ are the opposite Borel subalgebras corresponding to a choice $\{\alpha_i\}_{i\in\mathbf{I}}$ of simple roots of $\mathfrak{g}$. For any $i \in \mathbf{I}$, let $\mathfrak{sl}_2^i \subset \mathfrak{g}$ be the corresponding 3-dimensional subalgebra and denote by

$$U_\hbar(L\mathfrak{sl}_2^i) \subset U_\hbar(L\mathfrak{g}) \qquad \text{and} \qquad Y_\hbar(\mathfrak{sl}_2^i) \subset Y_\hbar(\mathfrak{g})$$

the subalgebras which deform $U(\mathfrak{sl}_2^i[z,z^{-1}])$ and $U(\mathfrak{sl}_2^i[s])$ respectively. Then, the main result of this paper is the following

Theorem *There exists an explicit algebra homomorphism $\Phi: U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ with the following properties*

(1) Φ is defined over $\mathbb{Q}[[\hbar]]$.
(2) Φ induces an isomorphism $\widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$, where $\widehat{U_\hbar(L\mathfrak{g})}$ is the completion of $U_\hbar(L\mathfrak{g})$ with respect to the ideal of $z=1$.
(3) Φ induces Drinfeld's degeneration of $U_\hbar(L\mathfrak{g})$ to $Y_\hbar(\mathfrak{g})$.
(4) Φ restricts to a homomorphism $U_\hbar(L\mathfrak{h}) \to \widehat{Y_\hbar(\mathfrak{h})}$ which induces the exponentiation of roots on Drinfeld polynomials.
(5) Φ restricts to a homomorphism $U_\hbar(L\mathfrak{b}_\pm) \to \widehat{Y_\hbar(\mathfrak{b}_\pm)}$.
(6) Φ restricts to a homomorphism $U_\hbar(L\mathfrak{sl}_2^i) \to \widehat{Y_\hbar(\mathfrak{sl}_2^i)}$ for any $i \in \mathbf{I}$.

It is interesting to note that Theorem 1.4 stands in stark contrast with the analogous finite-dimensional situation. Indeed, if $\mathfrak{g} \not\cong \mathfrak{sl}_2$, no explicit isomorphisms are known between the quantum group $U_\hbar\mathfrak{g}$ and the undeformed enveloping algebra $U\mathfrak{g}[[\hbar]]$ [5, §6.4]. Moreover, if $\mathfrak{g} \not\cong \mathfrak{sl}_2$, no algebra isomorphism $U_\hbar\mathfrak{g} \to U\mathfrak{g}[[\hbar]]$ maps $U_\hbar\mathfrak{sl}_2^i$ to $U\mathfrak{sl}_2^i[[\hbar]]$ for every $i \in \mathbf{I}$ [31, Prop. 3.2].

1.5

The homomorphism Φ has the following form. Let $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i\in\mathbf{I},k\in\mathbb{Z}}$ be the loop generators of $U_\hbar(L\mathfrak{g})$ and $\{x_{i,m}^\pm, \xi_{i,m}\}_{i\in\mathbf{I},m\in\mathbb{N}}$ those of $Y_\hbar(\mathfrak{g})$ (see [11] and Sect. 2 for definitions). Then,

$$\Phi(H_{i,0}) = d_i^{-1} t_{i,0}$$
$$\Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{m\geq 0} t_{i,m} \frac{r^m}{m!}$$
$$\Phi(E_{i,k}) = e^{k\sigma_i^+} \sum_{m\geq 0} g_{i,m}^+ \, x_{i,m}^+$$
$$\Phi(F_{i,k}) = e^{k\sigma_i^-} \sum_{m\geq 0} g_{i,m}^- \, x_{i,m}^-$$

In the formulae above, $r \in \mathbb{Z}^*$, $k \in \mathbb{Z}$, $q = e^{\hbar/2}$ and $q_i = q^{d_i}$, where the d_i are the symmetrising integers for the Cartan matrix of $\mathfrak{g}$. The $\{t_{i,m}\}_{i\in\mathbf{I},m\in\mathbb{N}}$ are an alternative set of generators of the commutative subalgebra $Y_\hbar(\mathfrak{h}) \subset Y_\hbar(\mathfrak{g})$ generated by the elements $\{\xi_{i,m}\}_{i\in\mathbf{I},m\in\mathbb{N}}$. They are defined in [22] by equating the generating functions

$$\hbar \sum_{m\geq 0} t_{i,m} u^{-m-1} = \log(1 + \hbar \sum_{m\geq 0} \xi_{i,m} u^{-m-1})$$

The elements $\{g_{i,m}^{\pm}\}_{i\in\mathbf{I},m\in\mathbb{N}}$ lie in the completion of $Y_\hbar(\mathfrak{h})$ and are constructed as follows. Consider the formal power series

$$G(v) = \log\left(\frac{v}{e^{v/2} - e^{-v/2}}\right) \in v\mathbb{Q}[[v]]$$

and define $\gamma_i(v) \in \widehat{Y^0}[[v]]$ by

$$\gamma_i(v) = \hbar \sum_{r\geq 0} \frac{t_{i,r}}{r!}\left(-\frac{d}{dv}\right)^{r+1} G(v)$$

Then,

$$\sum_{m\geq 0} g_{i,m}^{\pm} v^m = \left(\frac{\hbar}{q_i - q_i^{-1}}\right)^{1/2} \exp\left(\frac{\gamma_i(v)}{2}\right) \tag{1.1}$$

Finally, $\sigma_i^{\pm}$ are the homomorphisms of the subalgebras $Y_\hbar(\mathfrak{b}_{\pm}) \subset Y_\hbar(\mathfrak{g})$ generated by $\{\xi_{j,r}, x_{j,r}^{\pm}\}_{j\in\mathbf{I},r\in\mathbb{N}}$, which fix the $\xi_{j,r}$ and act on the remaining generators as the shifts $x_{j,r}^{\pm} \to x_{j,r+\delta_{ij}}^{\pm}$.

Note that the formulae connecting the generators $\{H_{i,k}\}$ of $U_\hbar(L\mathfrak{g})$ and $\{t_{i,m}\}$ of $Y_\hbar(\mathfrak{h})$ essentially coincide with those connecting the generators of $\mathfrak{h}[z, z^{-1}]$ and $\mathfrak{h}[s]$.

1.6

The above formulae apply equally well when $\mathfrak{g}$ is a symmetrisable Kac–Moody algebra. Our proof of Theorem 1.4 shows that they define a homomorphism from the quantum affinization $\widehat{U_\hbar\mathfrak{g}}$ of the quantum group $U_\hbar\mathfrak{g}$ [20,26], to the completion of the Yangian $Y_\hbar(\mathfrak{g})$, provided the following holds

(1) the entries of the Cartan matrix of $\mathfrak{g}$ satisfy $a_{ij}a_{ji} \leq 3$ for $i \neq j \in \mathbf{I}$.
(2) the PBW theorem holds for $Y_\hbar(\mathfrak{g})$.

The first assumption is equivalent to requiring that all rank 2 subalgebras of $\mathfrak{g}$ be finite-dimensional and is needed in our proof of the q-Serre relations. The second is required for the construction of certain straightening homomorphisms on $Y_\hbar(\mathfrak{g})$ which are needed in the proof of Theorem 1.4. We note that, for the Yangians associated with affine Kac–Moody algebras, the PBW theorem was proved by Guay in type A_n, for $n \geq 4$ [16] and, more recently, by Guay–Nakajima for all simply laced cases [18]. In particular, the above formulae define a homomorphism from the quantum toroïdal algebra $U_\hbar^{tor}(\mathfrak{g})$ associated with a simply laced, simple Lie algebra $\mathfrak{g} \not\cong \mathfrak{sl}_2$, to the completion of the affine Yangian $Y_\hbar\widehat{\mathfrak{g}}$.

1.7

We also construct in this paper a homomorphism similar to the one described in Sect. 1.5 for $\mathfrak{g} = \mathfrak{gl}_n$, by relying on the geometric realisation of $U_\hbar(L\mathfrak{gl}_n)$ obtained by Ginzburg and Vasserot [15,34]. More precisely, fix integers $1 \leq n \leq d$, and let

$$\mathcal{F} = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^d\}$$

be the variety of n-step flags in $\mathbb{C}^d$. The cotangent bundle $T^*\mathcal{F}$ may be realised as

$$T^*\mathcal{F} = \{(V_\bullet, x) \in \mathcal{F} \times \mathrm{End}(\mathbb{C}^d) |\, x(V_i) \subset V_{i-1}\}$$

and therefore admits a morphism $T^*\mathcal{F} \to \mathcal{N}$ via the second projection, where $\mathcal{N} = \{x \in \mathrm{End}(\mathbb{C}^d) |\, x^n = 0\}$ is the cone of n-step nilpotent endomorphisms. Define the Steinberg variety $Z = T^*\mathcal{F} \times_{\mathcal{N}} T^*\mathcal{F}$. The group $GL_d \times \mathbb{C}^\times$ acts on $T^*\mathcal{F}$ and Z and there are surjective algebra homomorphisms

$$\Psi_U : U_\hbar(L\mathfrak{gl}_n) \to K^{GL_d \times \mathbb{C}^\times}(Z)$$
$$\Psi_Y : Y_\hbar(\mathfrak{gl}_n) \to H^{GL_d \times \mathbb{C}^\times}(Z)$$

see [15,34] for the definition of Ψ_U.

To understand these more explicitly, one can use the convolution actions of $K^{GL_d \times \mathbb{C}^\times}(Z)$ on $K^{GL_d \times \mathbb{C}^\times}(T^*\mathcal{F})$ and of $H^{GL_d \times \mathbb{C}^\times}(Z)$ on $H_{GL_d \times \mathbb{C}^\times}(T^*\mathcal{F})$, which are faithful. By using the equivariant Chern character, we construct an algebra homomorphism

$$\Phi : U_\hbar(L\mathfrak{gl}_n) \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$$

which intertwines these two actions.

1.8

In the sequel to this paper [13], we shall prove that, for $\mathfrak{g}$ semisimple, a modification of the pull-back functor Φ^* converges for numerical values of $\hbar$ and defines a functor

$$\mathrm{Rep}_{\mathrm{fd}}(Y_a(\mathfrak{g})) \to \mathrm{Rep}_{\mathrm{fd}}(U_\epsilon(L\mathfrak{g}))$$

where $Y_a(\mathfrak{g})$ is the specialisation of $Y_\hbar(\mathfrak{g})$ at $\hbar = a \in \mathbb{C}\setminus\mathbb{R}$ and $\epsilon = \exp(\pi i a)$ and defines an equivalence of an explicit subcategory of $\mathrm{Rep}_{\mathrm{fd}}(Y_a\mathfrak{g})$ with $\mathrm{Rep}_{\mathrm{fd}}(U_\epsilon(L\mathfrak{g}))$.

1.9

It is worth pointing out that most of our results relating $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ have analogues for the affine and degenerate affine Hecke algebras $\mathcal{H}$ and $\mathcal{H}'$ associated with an affine Weyl group W [24] and were in fact inspired by these and their further study in [6]. Indeed, in [24], Lusztig constructs an explicit isomorphism between appropriate completions of $\mathcal{H}$ and $\mathcal{H}'$. In this context, the isomorphism can be understood in terms of, and in fact obtained from, the geometric realisations

$$\Xi : \mathcal{H} \xrightarrow{\sim} K^{G\times\mathbb{C}^\times}(Z)$$
$$\Xi' : \mathcal{H}' \xrightarrow{\sim} H^{G\times\mathbb{C}^\times}(Z)$$

where Z is the Steinberg variety corresponding to W [7,14].

1.10 Outline of the paper

In Sect. 2, we review the definition of the quantum loop algebra $U_\hbar(L\mathfrak{g})$ and Yangian $Y_\hbar(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$. We also introduce shift homomorphisms of the subalgebras $Y_\hbar(\mathfrak{b}_\pm)$, and straightening homomorphisms of the subalgebra $Y_\hbar(\mathfrak{h})$.

In Sect. 3, we consider assignments mapping the generators of $U_\hbar(L\mathfrak{g})$ to $\widehat{Y_\hbar(\mathfrak{g})}$. These have the form described in Sect. 1.5, where the elements $g_{i,m}^\pm \in \widehat{Y_\hbar(\mathfrak{h})}$ are, however, not necessarily given by formula (1.1). Our main result, Theorem 3.4, gives necessary and sufficient conditions for these elements to give rise to an algebra homomorphism. We call such homomorphisms *of geometric type* since, for $\mathfrak{g}$ simply laced, they are related to the Chern character in the geometric realisation described in Sect. 1.2.

The proof that the elements given by (1.1) satisfy the conditions of Theorem 3.4, and therefore give rise to an algebra homomorphism Φ, is given in Sect. 4 (Theorem 4.7). We also prove that the action of Φ on Drinfeld polynomials exponentiates their roots (Corollary 4.5).

In Sect. 5, we prove the essential uniqueness of homomorphisms of geometric type by showing that any two differ by conjugation by an element of the torus H and an invertible element of $\widehat{Y_\hbar(\mathfrak{h})}$ (Theorem 5.11).

In Sect. 6, we show that any homomorphism of geometric type Φ induces an isomorphism $\widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$, where $\widehat{U_\hbar(L\mathfrak{g})}$ is the completion with respect to the evaluation ideal at $z = 1$ (Theorem 6.2). We show moreover that the associated graded map coincides with Drinfeld's degeneration of $U_\hbar(L\mathfrak{g})$ to $Y_\hbar(\mathfrak{g})$ (Proposition 6.5).

Section 7 contains similar results for $\mathfrak{g} = \mathfrak{gl}_n$. In addition to constructing an explicit homomorphism $\Phi : U_\hbar(L\mathfrak{gl}_n) \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$ (Theorem 7.6), we review the geometric realisations of these algebras and show that Φ intertwines them (Theorem 7.19).

An appendix (Sect. 8) contains a proof of the Serre relations which is required to complete the proof of Theorem 3.4.

2 Quantum loop algebras and Yangians

2.1

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra and $(\cdot,\cdot)$ a non-degenerate, invariant bilinear form on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ a basis of simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ the entries of the corresponding Cartan matrix $\mathbf{A}$. Set $d_i = (\alpha_i, \alpha_i)/2$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in \mathbf{I}$. Let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the isomorphism determined by the inner product $(\cdot, \cdot)$ and set $h_i = \nu^{-1}(\alpha_i)/d_i$. Choose root vectors $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = h_i$. Recall that $\mathfrak{g}$ is presented on generators $\{e_i, f_i, h_i\}$ subject to the relations

$$[h_i, h_j] = 0$$

$$[h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} h_i$$

for any $i, j \in \mathbf{I}$ and, for any $i \neq j \in \mathbf{I}$

$$\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0$$
$$\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$$

A closely related, but slightly less standard presentation may be obtained by setting $t_i = \nu^{-1}(\alpha_i) = d_i h_i$ and choosing, for any $i \in \mathbf{I}$, root vectors $x_i^\pm \in \mathfrak{g}_{\pm\alpha_i}$ such that $[x_i^+, x_i^-] = t_i$. Then $\mathfrak{g}$ is presented on $\{x_i^\pm, t_i\}_{i\in\mathbf{I}}$ subject to the relations

$$[t_i, t_j] = 0$$

$$[t_i, x_j^\pm] = \pm d_i a_{ij} x_j^\pm$$

$$[x_i^+, x_j^-] = \delta_{ij} t_i$$

$$\mathrm{ad}(x_i^\pm)^{1-a_{ij}} x_j^\pm = 0$$

2.2

Throughout this paper, q and $\hbar$ are formal variables related by $q^2 = e^\hbar$. For any $i \in \mathbf{I}$, we set $q_i = q^{d_i} = e^{\hbar d_i/2}$. We use the standard notation for Gaussian integers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$
$$[n]_q! = [n]_q\,[n-1]_q \ldots [1]_q \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

2.3 The quantum loop algebra [11]

Let $U_\hbar(L\mathfrak{g})$ be the unital, associative algebra over $\mathbb{C}[[\hbar]]$ topologically generated by elements $\{E_{i,k}, F_{i,k}, H_{i,k}\}_{i\in\mathbf{I},k\in\mathbb{Z}}$ subject to the following relations

(QL1) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{Z}$

$$[H_{i,r}, H_{j,s}] = 0$$

(QL2) For any $i, j \in \mathbf{I}$ and $k \in \mathbb{Z}$,

$$[H_{i,0}, E_{j,k}] = a_{ij} E_{j,k} \qquad [H_{i,0}, F_{j,k}] = -a_{ij} F_{j,k}$$

(QL3) For any $i, j \in \mathbf{I}$ and $r \in \mathbb{Z}^\times$,

$$[H_{i,r}, E_{j,k}] = \frac{[ra_{ij}]_{q_i}}{r} E_{j,r+k} \qquad [H_{i,r}, F_{j,k}] = -\frac{[ra_{ij}]_{q_i}}{r} F_{j,r+k}$$

(QL4) For $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$

$$E_{i,k+1} E_{j,l} - q_i^{a_{ij}} E_{j,l} E_{i,k+1} = q_i^{a_{ij}} E_{i,k} E_{j,l+1} - E_{j,l+1} E_{i,k}$$
$$F_{i,k+1} F_{j,l} - q_i^{-a_{ij}} F_{j,l} F_{i,k+1} = q_i^{-a_{ij}} F_{i,k} F_{j,l+1} - F_{j,l+1} F_{i,k}$$

(QL5) For $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$

$$[E_{i,k}, F_{j,l}] = \delta_{ij} \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}$$

(QL6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For every $k_1, \ldots, k_m \in \mathbb{Z}$ and $l \in \mathbb{Z}$

$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} E_{i,k_{\pi(1)}} \ldots E_{i,k_{\pi(s)}} E_{j,l} E_{i,k_{\pi(s+1)}} \ldots E_{i,k_{\pi(m)}} = 0$$
$$\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} F_{i,k_{\pi(1)}} \ldots F_{i,k_{\pi(s)}} F_{j,l} F_{i,k_{\pi(s+1)}} \ldots F_{i,k_{\pi(m)}} = 0$$

where the elements $\psi_{i,r}, \phi_{i,r}$ are defined by

$$\psi_i(z) = \sum_{r \geq 0} \psi_{i,r} z^{-r} = \exp\left(\frac{\hbar d_i}{2} H_{i,0} \right) \exp\left((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s} \right)$$
$$\phi_i(z) = \sum_{r \geq 0} \phi_{i,-r} z^{r} = \exp\left(-\frac{\hbar d_i}{2} H_{i,0} \right) \exp\left(-(q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,-s} z^{s} \right)$$

with $\psi_{i,-k} = \phi_{i,k} = 0$ for every $k \geq 1$.

We shall denote by $U^0 \subset U_\hbar(L\mathfrak{g})$ the commutative subalgebra generated by the elements $\{H_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{Z}}$.

2.4 The Yangian [11]

Let $Y_\hbar(\mathfrak{g})$ be the unital, associative $\mathbb{C}[\hbar]$-algebra generated by elements $\{x_{i,r}^{\pm}, \xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$, subject to the following relations

(Y1) For any $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$

$$[\xi_{i,r}, \xi_{j,s}] = 0$$

(Y2) For $i, j \in \mathbf{I}$ and $s \in \mathbb{N}$

$$[\xi_{i,0}, x_{j,s}^{\pm}] = \pm d_i a_{ij} x_{j,s}^{\pm}$$

(Y3) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$

$$[\xi_{i,r+1}, x_{j,s}^{\pm}] - [\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm\frac{d_i a_{ij}\hbar}{2}(\xi_{i,r}x_{j,s}^{\pm} + x_{j,s}^{\pm}\xi_{i,r})$$

(Y4) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{d_i a_{ij}\hbar}{2}(x_{i,r}^{\pm}x_{j,s}^{\pm} + x_{j,s}^{\pm}x_{i,r}^{\pm})$$

(Y5) For $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$

$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij}\xi_{i,r+s}$$

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \ldots, r_m \in \mathbb{N}$ and $s \in \mathbb{N}$

$$\sum_{\pi\in\mathfrak{S}_m}\left[x_{i,r_{\pi(1)}}^{\pm}, \left[x_{i,r_{\pi(2)}}^{\pm}, \ldots, \left[x_{i,r_{\pi(m)}}^{\pm}, x_{j,s}^{\pm}\right]\cdots\right]\right] = 0$$

$Y_\hbar(\mathfrak{g})$ is an $\mathbb{N}$-graded algebra by $\deg(\xi_{i,r}) = \deg(x_{i,r}^{\pm}) = r$ and $\deg(\hbar) = 1$.

2.5 PBW theorem for $Y_\hbar(\mathfrak{g})$

For any positive root β of $\mathfrak{g}$, choose a sequence of simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ such that $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$ and

$$[x_{i_1}^{\pm}, [x_{i_2}^{\pm}, \ldots, [x_{i_{k-1}}^{\pm}, x_{i_k}^{\pm}]\cdots]] \in \mathfrak{g}_{\pm\beta}$$

are non-zero vectors. For any $r \in \mathbb{N}$, define $x_{\beta,r}^{\pm} \in Y_\hbar(\mathfrak{g})$ by choosing a partition $r = r_1 + \cdots + r_k$ of length k and setting

$$x_{\beta,r}^{\pm} = [x_{i_1,r_1}^{\pm}, [x_{i_2,r_2}^{\pm}, \ldots, [x_{i_{k-1},r_{k-1}}^{\pm}, x_{i_k,r_k}^{\pm}]\cdots]]$$

Theorem ([23]). *Fix a total order on the set* $\mathcal{G} = \{\xi_{i,r}, x^{\pm}_{\beta,r}\}_{i\in\mathbf{I},r\in\mathbb{N},\beta\in\Sigma_+}$. *Then, the ordered monomials in the elements of* $\mathcal{G}$ *form a basis of* $Y_\hbar(\mathfrak{g})$.

Let $Y^0, Y^\pm \subset Y_\hbar(\mathfrak{g})$ be the subalgebras generated by the elements $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ (resp. $\{x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$) and $Y^{\geq 0}, Y^{\leq 0} \subset Y_\hbar(\mathfrak{g})$ the subalgebras generated by Y^0, Y^+ and Y^0, Y^- respectively. The following is a direct consequence of Theorem 2.5.

Corollary (1) *Y^0 is a polynomial algebra in the generators $\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$.*
(2) *$Y^\pm$ is the algebra generated by elements $\{x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ subject to the relations (Y4) and (Y6).*
(3) *$Y^{\geq 0}$ (resp. $Y^{\leq 0}$) is the algebra generated by elements $\{\xi_{i,r}, x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ subject to the relations (Y1)–(Y4) and (Y6).*
(4) *Multiplication induces an isomorphism of vector spaces*

$$Y^- \otimes Y^0 \otimes Y^+ \to Y_\hbar(\mathfrak{g})$$

2.6 The shift operators $\sigma_i^\pm$

Fix $i \in \mathbf{I}$. By Corollary 2.5 (3), the assignment

$$x^\pm_{j,r} \to x^\pm_{j,r+\delta_{ij}} \qquad \xi_{j,r} \to \xi_{j,r}$$

extends to an algebra homomorphism $Y^{\geq 0} \to Y^{\geq 0}$ (resp. $Y^{\leq 0} \to Y^{\leq 0}$) which we shall denote by $\sigma_i^\pm$.

2.7 The relations (Y2)–(Y3)

We rewrite below the defining relations (Y2)–(Y3) of $Y_\hbar(\mathfrak{g})$ in terms of the shift operators $\sigma_j^\pm$ and the generating series

$$\xi_i(u) = 1 + \hbar \sum_{r\geq 0} \xi_{i,r} u^{-r-1} \in Y_\hbar(\mathfrak{g})[[u^{-1}]] \tag{2.1}$$

Lemma *The relations (Y2)–(Y3) are equivalent to*

$$[\xi_i(u), x^\pm_{j,s}] = \frac{\pm\hbar d_i a_{ij}}{u - \sigma_j^\pm \pm \hbar d_i a_{ij}/2} \xi_i(u) x^\pm_{j,s}$$

where the rational function on the right-hand side is expanded in powers of u^{-1}.

Proof Set $a = d_i a_{ij}/2$. Multiplying (Y3) by $\hbar u^{-r-1}$ and summing over $r \geq 0$ yields

$$u[\xi_i(u) - 1 - \hbar u^{-1}\xi_{i,0}, x^\pm_{j,s}] - [\xi_i(u) - 1, x^\pm_{j,s+1}] = \pm\hbar a\{x^\pm_{j,s}, \xi_i(u) - 1\}$$

where $\{x,\xi\} = x\xi + \xi x$. Using (Y2) and $\{x,\xi\} = [x,\xi] + 2\xi x$, yields

$$(u - \sigma_j^{\pm} \pm \hbar a)[\xi_i(u), x_{j,s}^{\pm}] = \pm 2\hbar a \xi_i(u) x_{j,s}^{\pm} \tag{2.2}$$

as claimed. Conversely, taking the coefficients of u^0 and u^{-r-1} in (2.2) yields (Y2) and (Y3) respectively. □

2.8 The relations (Y4) and (Y6)

We shall use the following notation

- for an operator $T \in \mathrm{End}(V)$, $T_{(i)} \in \mathrm{End}(V^{\otimes m})$ is defined as

$$T_{(i)} = 1^{\otimes i-1} \otimes T \otimes 1^{\otimes m-i}$$

- for an algebra A, $ad^{(m)} : A^{\otimes m} \to \mathrm{End}(A)$ is defined as

$$ad^{(m)}(a_1 \otimes \cdots \otimes a_m) = ad(a_1) \circ \cdots \circ ad(a_m)$$

Proposition (1) *The relation (Y4) for $i \neq j$ is equivalent to the requirement that the following holds for any $A(v_1, v_2) \in \mathbb{C}[[v_1, v_2]]$*

$$A(\sigma_i^{\pm}, \sigma_j^{\pm})(\sigma_i^{\pm} - \sigma_j^{\pm} \mp a\hbar) x_{i,0}^{\pm} x_{j,0}^{\pm} = A(\sigma_i^{\pm}, \sigma_j^{\pm})(\sigma_i^{\pm} - \sigma_j^{\pm} \pm a\hbar) x_{j,0}^{\pm} x_{i,0}^{\pm}$$

where $a = d_i a_{ij}/2$.

(2) *The relation (Y4) for $i = j$ is equivalent to the requirement that the following holds for any $B(v_1, v_2) \in \mathbb{C}[[v_1, v_2]]$ such that $B(v_1, v_2) = B(v_2, v_1)$*

$$\mu\left(B(\sigma_{i,(1)}^{\pm}, \sigma_{i,(2)}^{\pm})(\sigma_{i,(1)}^{\pm} - \sigma_{i,(2)}^{\pm} \mp d_i\hbar) x_{i,0}^{\pm} \otimes x_{i,0}^{\pm}\right) = 0 \tag{2.3}$$

where $\mu : Y_\hbar(\mathfrak{g})^{\otimes 2} \to Y_\hbar(\mathfrak{g})$ is the multiplication.

(3) *The relation (Y6) is equivalent to the requirement that the following holds for any $i \neq j$ and $A \in \mathbb{C}[v_1, \ldots, v_m]^{\mathfrak{S}_m}$ with $m = 1 - a_{ij}$*

$$ad^{(m)}\left(A(\sigma_{i,(1)}^{\pm}, \ldots, \sigma_{i,(m)}^{\pm})\left(x_{i,0}^{\pm}\right)^{\otimes m}\right) x_{j,l}^{\pm} = 0$$

Proof (1) The relation (Y4)

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm a\hbar(x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm})$$

may be rewritten as

$$\sigma_i^{\pm r} \sigma_j^{\pm s}\left(\sigma_i^{\pm} - \sigma_j^{\pm} \mp a\hbar\right) x_{i,0}^{\pm} x_{j,0}^{\pm} = \sigma_i^{\pm r} \sigma_j^{\pm s}\left(\sigma_i^{\pm} - \sigma_j^{\pm} \pm a\hbar\right) x_{j,0}^{\pm} x_{i,0}^{\pm}$$

(2) If $i=j$, $a=d_i$ and the above may be written as

$$\mu\left(\sigma^{\pm r}_{i,(1)}\sigma^{\pm s}_{i,(2)}\left(\sigma^{\pm}_{i,(1)}-\sigma^{\pm}_{i,(2)}\mp d_i\hbar\right)x^{\pm}_{i,0}\otimes x^{\pm}_{i,0}\right)$$
$$=\mu\left(\sigma^{\pm s}_{i,(1)}\sigma^{\pm r}_{i,(2)}\left(\sigma^{\pm}_{i,(2)}-\sigma^{\pm}_{i,(1)}\pm d_i\hbar\right)x^{\pm}_{i,0}\otimes x^{\pm}_{i,0}\right)$$

which is equivalent to

$$\mu\left(\left(\sigma^{\pm r}_{i,(1)}\sigma^{\pm s}_{i,(2)}+\sigma^{\pm s}_{i,(1)}\sigma^{\pm r}_{i,(2)}\right)\left(\sigma^{\pm}_{i,(1)}-\sigma^{\pm}_{i,(2)}\mp d_i\hbar\right)x^{\pm}_{i,0}\otimes x^{\pm}_{i,0}\right)=0$$

(3) is just the reformulation of (Y6).

□

Corollary *If* (2.3) *holds for some* $B\in\mathbb{C}[[v_1,v_2]]$, *then* $B(v_1,v_2)=B(v_2,v_1)$.

Proof By (2) of Proposition 2.8, we may assume that $B(v_1,v_2)=-B(v_2,v_1)$ and therefore that $B=(v_1-v_2)\overline{B}$ where $\overline{B}$ is symmetric in $v_1\leftrightarrow v_2$. Using the grading on $Y_\hbar(\mathfrak{g})$, we may further assume that $\overline{B}$ is proportional to $v_1^r v_2^s+v_1^s v_2^r$ for some $r\geq s\in\mathbb{N}$. An application of (Y4) then yields

$$\mu\Big((\sigma^{\pm}_{i,(1)}-\sigma^{\pm}_{i,(2)})(\sigma^{\pm r}_{i,(1)}\sigma^{\pm s}_{i,(2)}+\sigma^{\pm s}_{i,(1)}\sigma^{\pm r}_{i,(2)})(\sigma^{\pm}_{i,(1)}-\sigma^{\pm}_{i,(2)}\mp d_i\hbar)x^{\pm}_{i,0}\otimes x^{\pm}_{i,0}\Big)$$
$$=2\Big((x^{\pm}_{i,r+2}x^{\pm}_{i,s}-x^{\pm}_{i,r+1}x^{\pm}_{i,s+1}\mp d_i\hbar x^{\pm}_{i,r+1}x^{\pm}_{i,s})-(x^{\pm}_{i,r+1}x^{\pm}_{i,s+1}-x^{\pm}_{i,r}x^{\pm}_{i,s+2}\mp d_i\hbar x^{\pm}_{i,r}x^{\pm}_{i,s+1})\Big)$$

If $r\geq s+2$, the above is not zero by the PBW Theorem 2.5 and $\overline{B}=0$. If $r=s+1$, a further application of (Y4) shows that the second of the above two parenthesized summands is zero and again $\overline{B}=0$ by Theorem 2.5. Finally, if $r=s$, (Y4) implies that the two parenthesized summands are opposites of each other and again $\overline{B}=0$. □

2.9 An alternative system of generators for Y^0

The following generators of Y^0 were introduced in [22]. For any $i\in\mathbf{I}$, define the formal power series

$$t_i(u)=\hbar\sum_{r\geq 0}t_{i,r}u^{-r-1}\in Y^0[[u^{-1}]]$$

by

$$t_i(u)=\log(\xi_i(u))=\log\left(1+\hbar\sum_{r\geq 0}\xi_{i,r}u^{-r-1}\right) \tag{2.4}$$

Since (2.4) can be inverted, $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ is another system of generators of Y^0. These are homogeneous, with $\deg(t_{i,r})=r$, since $\zeta\cdot\xi_i(u)=\xi_i(\zeta^{-1}u)$, where $\cdot$ is the action of $\zeta\in\mathbb{C}^*$ on $Y_\hbar(\mathfrak{g})$ given by the grading. Moreover, $t_{i,0}=\xi_{i,0}$ and $t_{i,r}=\xi_{i,r}\mod\hbar$ for any $r\geq 1$ since

$$t_i(u) = \hbar \sum_{r\geq 0} \xi_{i,r} u^{-r-1} \mod \hbar^2$$

To compute the commutation relations between $t_{i,r}$ and $x^{\pm}_{j,s}$, we introduce the following formal power series (inverse Borel transform of $t_i(u)$)

$$B_i(v) = B(t_i(u)) = \hbar \sum_{r\geq 0} t_{i,r} \frac{v^r}{r!} \in Y^0[[v]] \tag{2.5}$$

Lemma *For any* $i, j \in \mathbf{I}$ *we have*

$$\left[B_i(v), x^{\pm}_{j,s}\right] = \pm \frac{q_i^{a_{ij}v} - q_i^{-a_{ij}v}}{v} e^{\sigma^{\pm}_j v} x^{\pm}_{j,s} \tag{2.6}$$

Proof To simplify notations set $a = \pm\hbar d_i a_{ij}/2$, so that $\mathrm{e}^a = q_i^{\pm a_{ij}}$. By Lemma 2.7

$$\xi_i(u) x^{\pm}_{j,s} \xi_i(u)^{-1} = \frac{u - \sigma^{\pm}_j + a}{u - \sigma^{\pm}_j - a} x^{\pm}_{j,s}$$

so that

$$[t_i(u), x^{\pm}_{j,s}] = \log\left(\frac{1 - (\sigma^{\pm}_j - a)u^{-1}}{1 - (\sigma^{\pm}_j + a)u^{-1}}\right) x^{\pm}_{j,s}$$

Using

$$B\left(\log(1 - pu^{-1})\right) = \frac{1 - \mathrm{e}^{pv}}{v} \tag{2.7}$$

this yields

$$[B_i(v), x^{\pm}_{j,s}] = \left(\frac{1 - \mathrm{e}^{(\sigma^{\pm}_j - a)v}}{v} - \frac{1 - \mathrm{e}^{(\sigma^{\pm}_j + a)v}}{v}\right) x^{\pm}_{j,s} = \frac{\mathrm{e}^{av} - \mathrm{e}^{-av}}{v} \mathrm{e}^{\sigma^{\pm}_j v} x^{\pm}_{j,s}$$

as claimed. □

Remark Expanding the right-hand side of (2.6) as a power series in v yields the commutation relations

$$[t_{i,r}, x^{\pm}_{j,s}] = \pm d_i a_{ij} \sum_{l=0}^{\lfloor r/2 \rfloor} \binom{r}{2l} \frac{(\hbar d_i a_{ij}/2)^{2l}}{2l+1} x^{\pm}_{j,r+s-2l}$$

These relations were obtained in this form in [22, Lemma 1.4].

2.10 The operators $\lambda_i^{\pm}(v)$

We introduce below operators which straighten monomials of the form $x^{\pm}_{i,m}\xi$, $\xi \in Y^0$, into elements of $Y^0 \cdot Y^{\pm}$.

Proposition *There are operators $\{\lambda^{\pm}_{i;s}\}_{i\in\mathbf{I},s\in\mathbb{N}}$ on Y^0 such that the following holds*

(1) *For any $\xi \in Y^0$, the elements $\lambda^{\pm}_{i;s}(\xi) \in Y^0$ are uniquely determined by the requirement that, for any $m \in \mathbb{N}$,*

$$x^{\pm}_{i,m}\xi = \sum_{s\geq 0} \lambda^{\pm}_{i;s}(\xi) x^{\pm}_{i,m+s} \tag{2.8}$$

(2) *For any $\xi, \eta \in Y^0$,*

$$\lambda^{\pm}_{i;s}(\xi\eta) = \sum_{k+l=s} \lambda^{\pm}_{i;k}(\xi)\lambda^{\pm}_{i;l}(\eta) \tag{2.9}$$

(3) *The operator $\lambda_{i;s} : Y^0 \to Y^0$ is homogeneous of degree $-s$.*

(4) *Let $\lambda_i^{\pm}(v) : Y^0 \to Y^0[v]$ be given by*

$$\lambda_i^{\pm}(v)(\xi) = \sum_{s\geq 0} \lambda^{\pm}_{i;s}(\xi)\, v^s$$

and extend the $\mathbb{N}$-grading on Y^0 to $Y^0[v]$ by $\deg(v) = 1$. Then $\lambda_i^{\pm}(v)$ is an algebra homomorphism of degree 0.

(5) $\lambda^{\epsilon_1}_{i_1}(v_1)$ *and* $\lambda^{\epsilon_2}_{i_2}(v_2)$ *commute for any $i_1, i_2 \in \mathbf{I}$ and $\epsilon_1, \epsilon_2 \in \{\pm\}$.*

(6) *For any $i \in \mathbf{I}$,*

$$\lambda_i^{+}(v)\lambda_i^{-}(v) = \mathrm{id} = \lambda_i^{-}(v)\lambda_i^{+}(v)$$

(7) *For any $i, j \in \mathbf{I}$,*

$$\lambda_j^{\pm}(v_1)\,(B_i(v_2)) = B_i(v_2) \mp \frac{q_i^{a_{ij}v_2} - q_i^{-a_{ij}v_2}}{v_2} e^{v_1 v_2}$$

(8) *For any $i \in \mathbf{I}$ and $r \in \mathbb{N}$,*

$$\lambda_j^{\pm}(v)(t_{i,r}) = t_{i,r} \mp d_i a_{ij} v^r \mod \hbar$$

Proof (1)–(2) by Lemma 2.7, (2.8) holds when ξ is one of the generators $\xi_{j,r}$ of Y^0. Since (2.8) holds for $\xi\eta$ if it holds for $\xi, \eta \in Y^0$, with $\lambda_{i;s}(\xi\eta)$ given by (2.9), the $\lambda_{i;s}$ can be defined as operators on Y^0. The fact they are uniquely characterised by (2.8) and satisfy (2.9) follows from Corollary 2.5.

(3) the linear independence of the elements on the right-hand side of (2.8) implies that $\deg(\lambda_{i;s}(\xi)) = \deg(\xi) - s$ for any homogeneous $\xi \in Y^0$. (4) is a rephrasing of (2) and (3). (5) and (6) follow from (7) since the elements $\{t_{i,n}\}$ generate Y^0. (7) follows from Lemma 2.9. (8) is a direct consequence of (7). □

Remark Using the shift operators, the relation (2.8) can be rewritten as

$$x^{\pm}_{i,m}\xi = \lambda^{\pm}_i(\sigma^{\pm}_i)(\xi)\, x^{\pm}_{i,m}$$

3 Homomorphisms of geometric type

Let $\widehat{Y_\hbar\mathfrak{g}}$ be the completion of $Y_\hbar(\mathfrak{g})$ with respect to its $\mathbb{N}$-grading. In this section, we define an assignment

$$\Phi : \{H_{i,r}, E_{i,r}, F_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}} \longrightarrow \widehat{Y_\hbar\mathfrak{g}}$$

and find necessary and sufficient conditions for Φ to extend to a homomorphism $U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar\mathfrak{g}}$.

3.1 Definition of Φ

Define

$$\Phi(H_{i,0}) = d_i^{-1} t_{i,0}$$

and, for $r \in \mathbb{Z}^\times$

$$\Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k\geq 0} t_{i,k} \frac{r^k}{k!} = \frac{B_i(r)}{q_i - q_i^{-1}}$$

where $B_i(v)$ is the formal power series (2.5). Let $\widetilde{U}^0$ be the polynomial ring on the generators $\{H_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}}$.[1] The above assignment extends to an homomorphism $\Phi^0 : \widetilde{U}^0 \to \widehat{Y^0}$.

Let now $\{g^{\pm}_{i,m}\}_{i\in\mathbf{I},m\in\mathbb{N}}$ be elements of $\widehat{Y^0}$ and define further

$$\Phi(E_{i,0}) = \sum_{m\geq 0} g^{+}_{i,m} x^{+}_{i,m}$$

$$\Phi(F_{i,0}) = \sum_{m\geq 0} g^{-}_{i,m} x^{-}_{i,m}$$

[1] $\widetilde{U}^0$ is isomorphic to the subalgebra U^0 of $U_\hbar(L\mathfrak{g})$ generated by $\{H_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}}$ by the PBW Theorem for $U_\hbar(L\mathfrak{g})$ [1], but we shall not need this fact.

In terms of the shift operators $\sigma_i^{\pm}$, the above may be written as

$$\Phi(E_{i,0}) = g_i^+(\sigma_i^+)x_{i,0}^+ \tag{3.1}$$

$$\Phi(F_{i,0}) = g_i^-(\sigma_i^-)x_{i,0}^- \tag{3.2}$$

where

$$g_i^{\pm}(v) = \sum_{m\geq 0} g_{i,m}^{\pm} v^m \in \widehat{Y^0[v]}$$

with the completion of $Y^0[v]$ taken with respect to the $\mathbb{N}$-grading which extends that on Y^0 by $\deg(v) = 1$.

3.2 Homomorphisms of geometric type

If $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ is a homomorphism of the above form, we shall say that it is of *geometric type* since its form is related to the Chern character in the geometric realisations of $U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ discussed in Sect. 1.2.

Such a homomorphism has the following properties

(1) It restricts to a homomorphism $U_\hbar(L\mathfrak{sl}_2^i) \to \widehat{Y_\hbar(\mathfrak{sl}_2^i)}$ for any $\mathbf{I}$, where $U_\hbar(L\mathfrak{sl}_2^i) \subset U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{sl}_2^i) \subset Y_\hbar(\mathfrak{g})$ are the subalgebras generated by $\{E_{i,r}, F_{i,r}, H_{i,r}\}_{r\in\mathbb{Z}}$ and $\{x_{i,k}^{\pm}, \xi_{i,k}\}_{k\in\mathbb{N}}$ respectively.

(2) It restricts to a homomorphism $U_\hbar(L\mathfrak{b}_\pm) \to \widehat{Y_\hbar(\mathfrak{b}_\pm)}$, where $U_\hbar(L\mathfrak{b}_+), U_\hbar(L\mathfrak{b}_-) \subset U_\hbar(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{b}_\pm) \subset Y_\hbar(\mathfrak{g})$ are the subalgebras generated respectively by $\{E_{i,r}, H_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}}$, $\{F_{i,r}, H_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}}$ and $\{x_{i,k}^{\pm}, \xi_{i,k}\}_{i\in\mathbf{I},k\in\mathbb{N}}$.

Note however that, unless $g_{i,m}^{\pm} = 0$ for any $i \in \mathbf{I}$ and $m \geq 1$, Φ does not map the quantum group $U_\hbar\mathfrak{g} = \langle E_{i,0}, F_{i,0}, H_{i,0}\rangle_{i\in\mathbf{I}} \subset U_\hbar(L\mathfrak{g})$ to $U\mathfrak{g}[[\hbar]] \subset \widehat{Y_\hbar(\mathfrak{g})}$.

3.3

The following result shows that the requirement that Φ extends to an algebra homomorphism determines its value on generators $E_{i,k}, F_{i,k}$.

Proposition *The assignment Φ is compatible with relations (QL2)–(QL3) if, and only if*

$$\Phi(E_{i,k}) = e^{k\sigma_i^+} g_i^+(\sigma_i^+)x_{i,0}^+ \tag{3.3}$$

$$\Phi(F_{i,k}) = e^{k\sigma_i^-} g_i^-(\sigma_i^-)x_{i,0}^- \tag{3.4}$$

Proof We only consider the case of the $E's$. Let $i, j \in \mathbf{I}$ and $k \in \mathbb{Z}$. By (Y2),

$$\begin{aligned}[\Phi(H_{i,0}), \Phi(E_{j,k})] &= [d_i^{-1}\xi_{i,0}, \mathrm{e}^{k\sigma_j^+} g_j^+(\sigma_j^+)x_{j,0}^+] \\ &= \mathrm{e}^{k\sigma_j^+} g_j^+(\sigma_j^+)[d_i^{-1}\xi_{i,0}, x_{j,0}^+] \\ &= a_{ij}\Phi(E_{j,k})\end{aligned}$$

so that Φ is compatible with (QL2). Next, if $r \in \mathbb{Z}^\times$, Lemma 2.9 yields

$$\begin{aligned}[\Phi(H_{i,r}), \Phi(E_{j,k})] &= \frac{1}{q_i - q_i^{-1}}[B_i(r), \mathrm{e}^{k\sigma_j^+} g_j^+(\sigma_j^+)x_{j,0}^+] \\ &= \frac{q_i^{ra_{ij}} - q_i^{-ra_{ij}}}{r(q_i - q_i^{-1})}\mathrm{e}^{r\sigma_j^+}\mathrm{e}^{k\sigma_j^+} g_j^+(\sigma_j^+)x_{j,0}^+ \\ &= \frac{[ra_{ij}]_{q_i}}{r}\Phi(E_{j,r+k})\end{aligned}$$

and Φ is compatible with (QL3).

Conversely, if Φ is compatible with (QL3) then $\Phi(E_{i,r}) = r/[2r]_{q_i}[\Phi(H_{i,r}), \Phi(E_{i,0})]$ for $r \neq 0$ and the computation above shows that this is equal to $\mathrm{e}^{r\sigma_i^+}\Phi(E_{i,0})$. □

3.4 Necessary and sufficient conditions

Let $\lambda_i^\pm(v) : Y^0 \to Y^0[v]$ be the homomorphism defined in Proposition 2.10.

Theorem *The assignment Φ extends to an algebra homomorphism $U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ if, and only if the following conditions hold*

(A) *For any $i, j \in \mathbf{I}$*

$$g_i^+(u)\lambda_i^+(u)(g_j^-(v)) = g_j^-(v)\lambda_j^-(v)(g_i^+(u))$$

(B) *For any $i \in \mathbf{I}$ and $k \in \mathbb{Z}$*

$$e^{ku}g_i^+(u)\lambda_i^+(u)(g_i^-(u))\Big|_{u^m=\xi_{i,m}} = \Phi^0\left(\frac{\psi_{i,k} - \phi_{i,k}}{q_i - q_i^{-1}}\right)$$

(C) *For any $i, j \in \mathbf{I}$ and $a = d_i a_{ij}/2$*

$$g_i^\pm(u)\lambda_i^\pm(u)(g_j^\pm(v))\left(\frac{e^u - e^{v\pm a\hbar}}{u - v \mp a\hbar}\right) = g_j^\pm(v)\lambda_j^\pm(v)(g_i^\pm(u))\left(\frac{e^v - e^{u\pm a\hbar}}{v - u \mp a\hbar}\right)$$

Proof By construction and Proposition 3.3, Φ is compatible with the relations (QL1)–(QL3). The result then follows from Lemmas 3.5 and 3.6 below and the proof of the q-Serre relations (Proposition 8.1). □

3.5

Lemma *Φ is compatible with the relation (QL5) if, and only if (A) and (B) hold.*

Proof Compatibility with (QL5) reads

$$[\Phi(E_{i,k}), \Phi(F_{j,l})] = \delta_{ij}\Phi^0\left(\frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}\right)$$

for $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$. We begin by computing the left-hand side. To this end, it will be convenient to write formulae (3.3)–(3.4) as

$$\Phi(E_{i,k}) = \sum_{m\geq 0} g_{i,m}^{+,(k)}\, x_{i,m}^{+} \qquad \text{and} \qquad \Phi(F_{i,k}) = \sum_{m\geq 0} g_{i,m}^{-,(k)}\, x_{i,m}^{-}$$

where $g_{i,m}^{\pm,(k)} \in \widehat{Y^0}$ are defined by $\sum_{m\geq 0} g_{i,m}^{\pm,(k)} v^m = \mathrm{e}^{kv} g_i^{\pm}(v)$. This yields

$$\begin{aligned}
\Phi(E_{i,k})\Phi(F_{j,l}) &= \sum_{m,n\geq 0} g_{i,m}^{+,(k)} x_{i,m}^{+} g_{j,n}^{-,(l)} x_{j,n}^{-}\\
&= \sum_{m,n,s\geq 0} g_{i,m}^{+,(k)} \lambda_{i,s}^{+} \left(g_{j,n}^{-,(l)}\right) x_{i,m+s}^{+} x_{j,n}^{-}\\
&= \sum_{m,n,s\geq 0} g_{i,m}^{+,(k)} \lambda_{i,s}^{+} \left(g_{j,n}^{-,(l)}\right) \left(x_{j,n}^{-} x_{i,m+s}^{+} + \delta_{ij}\xi_{i,m+n+s}\right)
\end{aligned}$$

where we used (Y5). Similarly, $\Phi(F_{j,l})\Phi(E_{i,k}) = \sum_{m,n,s} g_{j,m}^{-,(l)} \lambda_{j,s}^{-} \left(g_{i,n}^{+,(k)}\right)$ $x_{j,m+s}^{-} x_{i,n}^{+}$. Define $R^{(k,l)}, L^{(k,l)} \in \widehat{Y^0}[[u, v]]$ by

$$R^{(k,l)} = \sum_{m\geq 0} g_{i,m}^{+,(k)} u^m \sum_{s\geq 0} \lambda_{i,s}^{+} u^s \left(\sum_{n\geq 0} g_{j,n}^{-,(l)} v^n\right) = \mathrm{e}^{ku}\mathrm{e}^{lv} g_i^{+}(u)\lambda_i^{+}(u)(g_j^{-}(v))$$

$$L^{(k,l)} = \sum_{m\geq 0} g_{j,m}^{-,(l)} v^m \sum_{s\geq 0} \lambda_{j,s}^{-} v^s \left(\sum_{n\geq 0} g_{i,n}^{+,(k)} u^n\right) = \mathrm{e}^{ku}\mathrm{e}^{lv} g_j^{-}(v)\lambda_j^{-}(v)(g_i^{+}(u))$$

By the PBW Theorem 2.5, Φ is compatible with (QL5) if, and only if $R^{(k,l)} = L^{(k,l)}$ and, for $i = j$,

$$R^{(k,l)}\Big|_{u^m v^n = \xi_{i,m+n}} = \Phi^0\left(\frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}\right)$$

The first equation is clearly equivalent to (A) and the second to (B). $\square$

3.6

Lemma Φ *is compatible with the relation (QL4) if, and only if (C) holds.*

Proof We prove the claim for the E's only. Compatibility with (QL4) reads

$$\Phi(E_{i,k+1})\Phi(E_{j,l}) - q_i^{a_{ij}}\Phi(E_{i,k})\Phi(E_{j,l+1}) = q_i^{a_{ij}}\Phi(E_{j,l})\Phi(E_{i,k+1}) - \Phi(E_{j,l+1})\Phi(E_{i,k})$$

for any $i, j \in \mathbf{I}$ and $k, l \in \mathbb{Z}$. Assume first $i \neq j$ and set $a = d_i a_{ij}/2$, so that $q_i^{a_{ij}} = \mathrm{e}^{a\hbar}$. Since

$$\Phi(E_{i,r})\Phi(E_{j,s}) = \mathrm{e}^{r\sigma_i^+}\mathrm{e}^{s\sigma_j^+} g_i^+(\sigma_i^+)\lambda_i^+(\sigma_i^+)\left(g_j^+(\sigma_j^+)\right) x_{i,0}^+ x_{j,0}^+$$

the above reduces to

$$\begin{aligned}&\mathrm{e}^{k\sigma_i^+}\mathrm{e}^{l\sigma_j^+} g_i^+(\sigma_i^+)\lambda_i^+(\sigma_i^+)(g_j^+(\sigma_j^+))\left(\mathrm{e}^{\sigma_i^+} - \mathrm{e}^{\sigma_j^+ + a\hbar}\right) x_{i,0}^+ x_{j,0}^+\\ &\quad = \mathrm{e}^{k\sigma_i^+}\mathrm{e}^{l\sigma_j^+} g_j^+(\sigma_j^+)\lambda_j^+(\sigma_j^+)(g_i^+(\sigma_i^+))\left(\mathrm{e}^{\sigma_i^+ + a\hbar} - \mathrm{e}^{\sigma_j^+}\right) x_{j,0}^+ x_{i,0}^+\end{aligned}$$

Using (1) of Proposition 2.8, we get

$$\begin{aligned}\left(\mathrm{e}^{\sigma_i^+} - \mathrm{e}^{\sigma_j^+ + a\hbar}\right) x_{i,0}^+ x_{j,0}^+ &= \frac{\mathrm{e}^{\sigma_i^+} - \mathrm{e}^{\sigma_j^+ + a\hbar}}{\sigma_i^+ - \sigma_j^+ - a\hbar}\left(\sigma_i^+ - \sigma_j^+ - a\hbar\right) x_{i,0}^+ x_{j,0}^+\\ &= \frac{\mathrm{e}^{\sigma_i^+} - \mathrm{e}^{\sigma_j^+ + a\hbar}}{\sigma_i^+ - \sigma_j^+ - a\hbar}\left(\sigma_i^+ - \sigma_j^+ + a\hbar\right) x_{j,0}^+ x_{i,0}^+\end{aligned}$$

The PBW Theorem 2.5 then shows that the above is equivalent to (C).

Assume now that $i = j$, then

$$\begin{aligned}\Phi(E_{i,r})\Phi(E_{i,s}) &= \left(\mathrm{e}^{r\sigma_i^+} g_i^+(\sigma_i^+) x_{i,0}^+\right)\left(\mathrm{e}^{s\sigma_i^+} g_i^+(\sigma_i^+) x_{i,0}^+\right)\\ &= \mu\left(\mathrm{e}^{r\sigma_{i,(1)}^+}\mathrm{e}^{s\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+)\lambda_i^+(\sigma_{i,(1)}^+)\left(g_i^+(\sigma_{i,(2)}^+)\right) x_{i,0}^+ \otimes x_{i,0}^+\right)\end{aligned}$$

The compatibility with (QL4) therefore reduces to

$$\begin{aligned}&\mu\left(\mathrm{e}^{k\sigma_{i,(1)}^+}\mathrm{e}^{l\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+)\lambda_i^+(\sigma_{i,(1)}^+)(g_i^+(\sigma_{i,(2)}^+))\left(\mathrm{e}^{\sigma_{i,(1)}^+} - \mathrm{e}^{\sigma_{i,(2)}^+ + d_i\hbar}\right) x_{i,0}^+ \otimes x_{i,0}^+\right)\\ &\quad = \mu\left(\mathrm{e}^{l\sigma_{i,(1)}^+}\mathrm{e}^{k\sigma_{i,(2)}^+} g_i^+(\sigma_{i,(1)}^+)\lambda_i^+(\sigma_{i,(1)}^+)(g_i^+(\sigma_{i,(2)}^+))\left(\mathrm{e}^{\sigma_{i,(2)}^+ + d_i\hbar} - \mathrm{e}^{\sigma_{i,(1)}^+}\right) x_{i,0}^+ \otimes x_{i,0}^+\right)\end{aligned}$$

that is, to

$$\begin{aligned}&\mu\Big(\left(\mathrm{e}^{k\sigma_{i,(1)}^+}\mathrm{e}^{l\sigma_{i,(2)}^+} + \mathrm{e}^{l\sigma_{i,(1)}^+}\mathrm{e}^{k\sigma_{i,(2)}^+}\right)\\ &g_i^+(\sigma_{i,(1)}^+)\lambda_i^+(\sigma_{i,(1)}^+)(g_i^+(\sigma_{i,(2)}^+))\left(\mathrm{e}^{\sigma_{i,(1)}^+} - \mathrm{e}^{\sigma_{i,(2)}^+ + d_i\hbar}\right) x_{i,0}^+ \otimes x_{i,0}^+\Big) = 0\end{aligned}$$

By (2) of Proposition 2.8 and Corollary 2.8, this equation is equivalent to the requirement that

$$g_i^+(u)\lambda_i^+(u)(g_i^+(v))\left(\frac{e^u - e^{v+d_i\hbar}}{u - v - d_i\hbar}\right)$$

be symmetric under $u \leftrightarrow v$, which is precisely condition (C) for $i = j$. □

3.7

For later use, we shall need the following

Lemma *Let* $\{g_i^\pm(u)\}_{i\in\mathbf{I}} \subset \widehat{Y^0[u]}$ *be elements satisfying condition (B) of Theorem 3.4. Then,*

$$g_i^\pm(u) = \frac{1}{d_i^\pm} \quad \mathrm{mod}\ \widehat{Y^0[u]}_+$$

where $\{d_i^\pm\}_{i\in\mathbf{I}} \subset \mathbb{C}^\times$ *satisfy* $d_i^+ d_i^- = d_i$ *for each* $i \in \mathbf{I}$. *In particular, each* $g_i^\pm(u)$ *is invertible.*

Proof Condition (B) for $k = 0$ yields

$$g_i^+(u)\lambda_i^+(u)\left(g_i^-(u)\right)\big|_{u^m=\xi_{i,m}} = \Phi^0\left(\frac{e^{\frac{\hbar d_i}{2}H_{i,0}} - e^{-\frac{\hbar d_i}{2}H_{i,0}}}{q_i - q_i^{-1}}\right)$$

Computing mod $\hbar$, and a fortiori mod $\widehat{Y^0[u]}_+$, yields

$$\Phi^0\left(\frac{e^{\frac{\hbar d_i}{2}H_{i,0}} - e^{-\frac{\hbar d_i}{2}H_{i,0}}}{q_i - q_i^{-1}}\right) = \Phi^0\left(H_{i,0}\right) = d_i^{-1}t_{i,0}$$

Write $g_i^\pm(u) = p_i^\pm \ \mathrm{mod}\ \widehat{Y^0[u]}_+$, where $p_i^\pm \in \mathbb{C}[t_{j,0}]_{j\in\mathbf{I}}$. Computing mod $\widehat{Y^0[u]}_+$, we get

$$g_i^+(u)\lambda_i^+(u)\left(g_i^-(u)\right)\Big|_{u^m=\xi_{i,m}} = p_i^+(t_{j,0})\lambda_{i;0}^+(p_i^-(t_{j,0}))\xi_{i,0} = p_i^+(t_{j,0})p_i^-(t_{j,0} - d_i a_{ij})\xi_{i,0}$$

where we used (8) of Proposition 2.10. Comparing both sides and using $\xi_{i,0} = t_{i,0}$ yields the claim. □

4 Existence of homomorphisms

In this section, we construct an explicit homomorphism $U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ by exhibiting a joint solution to equations (A)–(C) of Theorem 3.4. We begin by giving an intrinsic expression for the right-hand side of equation (B) (Proposition 4.2). Until 4.7, we fix $i \in \mathbf{I}$ and consider the subalgebras $\widetilde{U}_i^0 \subset \widetilde{U}^0$, $Y_i^0 \subset Y^0$ generated by $\{H_{i,r}\}_{r\in\mathbb{Z}}$ and $\{\xi_{i,r}\}_{r\in\mathbb{N}}$ respectively.

4.1 The functions $\mathbf{G}(\mathbf{v})$ and $\gamma_{\mathbf{i}}(\mathbf{v})$

Consider the formal power series

$$G(v) = \log\left(\frac{v}{\mathrm{e}^{v/2} - \mathrm{e}^{-v/2}}\right) \in v\mathbb{Q}[[v]] \tag{4.1}$$

Define $\gamma_i(v) \in \widehat{Y_i^0[v]}_+$ by

$$\gamma_i(v) = \hbar \sum_{r\geq 0} \frac{t_{i,r}}{r!} (-\partial_v)^{r+1}\, G(v)$$

Recall that $B_i(v) = \hbar \sum_{r\geq 0} t_{i,r} \frac{v^r}{r!}$ is the inverse Borel transform of $t_i(u)$. This allows us to write $\gamma_i(u)$ more compactly as

$$\gamma_i(v) = -B_i(-\partial_v)G'(v) \tag{4.2}$$

4.2

Proposition *The following holds in $\widehat{Y_i^0}$ for any $k \in \mathbb{Z}$*

$$\Phi^0\left(\frac{\psi_{i,k} - \phi_{i,k}}{q_i - q_i^{-1}}\right) = \frac{\hbar}{q_i - q_i^{-1}}\, e^{kv} \exp\left(\gamma_i(v)\right)\Big|_{v^n = \xi_{i,n}}$$

The above identity will be proved in Sects. 4.3–4.6 by injectively mapping both sides to a family of polynomial rings, and verifying their equality there.

4.3 Universal Drinfeld polynomials

Fix an integer $m \geq 1$. Following [26,33], consider the rings

$$S(m) = \mathbb{C}[q^{\pm 1}, A_1^{\pm 1}, \ldots, A_m^{\pm 1}]^{\mathfrak{S}_m}$$
$$R(m) = \mathbb{C}[\hbar, a_1, \ldots, a_m]^{\mathfrak{S}_m}$$

Define a homomorphism $\mathcal{D}^U : \widetilde{U}_i^0 \to S(m)$ by

$$\mathcal{D}^U(\psi_i(z)) = \prod_{p=1}^{m} \frac{q_i z - q_i^{-1} A_p}{z - A_p} = \mathcal{D}^U(\phi_i(z)) \tag{4.3}$$

where the first and second equalities are obtained by expanding the middle term in powers of z^{-1} and z respectively. Similarly, define $\mathcal{D}^Y : Y_i^0 \to R(m)$ by

$$\mathcal{D}^Y(\xi_i(u)) = \prod_{p=1}^{m} \frac{u + d_i\hbar - a_p}{u - a_p} \tag{4.4}$$

The homomorphism $\mathcal{D}^U$ (resp. $\mathcal{D}^Y$) gives the action of $\psi_i(z), \phi_i(z)$ (resp. $\xi_i(u)$) on the highest weight vector of the indecomposable $U_\hbar(L\mathfrak{sl}_2)$ (resp. $Y_\hbar(\mathfrak{sl}_2)$) module with Drinfeld polynomial $(1 - A_1^{-1}z)\dots(1 - A_m^{-1}z)$ (resp. $(u - a_1)\dots(u - a_m)$).

4.4

The following result spells out the image of the generators of $\widetilde{U}_i^0$ and Y_i^0 under $\mathcal{D}^U$ and $\mathcal{D}^Y$ respectively.

Proposition

(1) *The following holds:* $\mathcal{D}^U(\psi_{i,0}) = q_i^m = \mathcal{D}^U(\phi_{i,0})^{-1}$ *and, for any* $r \in \mathbb{N}^*$

$$\mathcal{D}^U(\psi_{i,r}) = \quad (q_i - q_i^{-1}) \sum_{p=1}^{m} A_p^r \prod_{p' \neq p} \frac{q_i A_p - q_i^{-1} A_{p'}}{A_p - A_{p'}} \tag{4.5}$$

$$\mathcal{D}^U(\phi_{i,-r}) = -(q_i - q_i^{-1}) \sum_{p=1}^{m} A_p^{-r} \prod_{p' \neq p} \frac{q_i A_p - q_i^{-1} A_{p'}}{A_p - A_{p'}} \tag{4.6}$$

Moreover, $\mathcal{D}^U$ *maps* $H_{i,0}$ *to* m *and, for any* $r \in \mathbb{Z}^*$,

$$\mathcal{D}^U((q_i - q_i^{-1})H_{i,r}) = \frac{1 - q_i^{-2r}}{r} \sum_{p=1}^{m} A_p^r \tag{4.7}$$

(2) *The homomorphism* $\mathcal{D}^Y$ *maps* $\xi_{i,0}$ *to* $d_i m$ *and, for any* $r \in \mathbb{N}$

$$\mathcal{D}^Y(\xi_r) = d_i \sum_{p=1}^{m} a_p^r \prod_{p' \neq p} \frac{a_p - a_{p'} + d_i\hbar}{a_p - a_{p'}} \tag{4.8}$$

Moreover, for any $r \in \mathbb{N}$,

$$\mathcal{D}^Y(t_r) = \frac{1}{r+1} \sum_{p=1}^{m} \frac{a_p^{r+1} - (a_p - d_i\hbar)^{r+1}}{\hbar} \tag{4.9}$$

(3) *If* $B_i(v) \in Y_i^0[[v]]$ *is the series defined by* (2.5), *then*

$$\mathcal{D}^Y(B_i(v)) = \frac{1 - e^{-d_i \hbar v}}{v} \sum_{p=1}^m e^{a_p v} \tag{4.10}$$

Proof

(1) The fact that $\mathcal{D}^U(\psi_{i,0}) = q_i^m = \mathcal{D}^U(\phi_{i,0})^{-1}$ follows by taking the values of the middle term $P(z)$ in (4.3) at $z = \infty$ and $z = 0$ respectively. Next, the partial fraction decomposition of $P(z)$ is readily seen to be

$$\prod_{p=1}^m \frac{q_i z - q_i^{-1} A_p}{z - A_p} = q_i^m + (q_i - q_i^{-1}) \sum_{p=1}^m A_p \left(\prod_{p' \neq p} \frac{q_i A_p - q_i^{-1} A_{p'}}{A_p - A_{p'}} \right) \frac{1}{z - A_p} \tag{4.11}$$

The relations (4.5)–(4.6) follow by expanding this into powers of z^{-1} and z respectively. For later use, note that the evaluation of (4.11) at $z = 0$ yields the identity

$$\mathcal{D}^U(\psi_{i,0} - \phi_{i,0}) = q_i^m - q_i^{-m} = (q_i - q_i^{-1}) \sum_{p=1}^m \prod_{p' \neq p} \frac{q_i A_p - q_i^{-1} A_{p'}}{A_p - A_{p'}} \tag{4.12}$$

Since $\mathcal{D}^U(\psi_{i,0}) = q_i^m$, it follows that $\mathcal{D}^U(H_{i,0}) = m$, and

$$\mathcal{D}^U\left(\exp\left((q_i - q_i^{-1}) \sum_{s \geq 1} H_{i,s} z^{-s}\right)\right) = \mathcal{D}^U(\psi_{i,0}^{-1} \psi_i(z)) = \prod_{p=1}^m \frac{z - q_i^{-2} A_p}{z - A_p}$$

taking the log of both sides and expanding in powers of z^{-1} (resp. z) yields (4.7) for $r > 0$ (resp. $r < 0$).

(2) The fact that $\mathcal{D}^Y(\xi_{i,0}) = d_i m$ follows by taking the coefficient of u^{-1} in (4.4). The partial fraction decomposition of $\mathcal{D}^Y(\xi_i(u))$ is

$$\prod_{p=1}^m \frac{u + d_i \hbar - a_p}{u - a_p} = 1 + d_i \hbar \sum_{p=1}^m \left(\prod_{p' \neq p} \frac{a_p - a_{p'} + d_i \hbar}{a_p - a_{p'}} \right) \frac{1}{u - a_p}$$

and (4.8) follows by taking the coefficient of u^{-r-1}. Taking the log of both sides of (4.4) yields

$$\mathcal{D}^Y(t_i(u)) = \sum_p -\log\left(1 - a_p u^{-1}\right) + \log\left(1 - (a_p - d_i \hbar) u^{-1}\right) \tag{4.13}$$

and therefore (4.9).

(3) Follows by applying (2.7) to (4.4). □

Corollary *The homomorphism* $\mathcal{D}^Y : Y_i^0 \to \bigoplus_{m\geq 1} R(m)$ *is injective.*

Proof This follows from (4.9) and the fact that the power sums $\sum_p a_p^r$ are algebraically independent. ◻

4.5

Let $\widehat{R(m)}$ be the completion of $R(m)$ with respect to the $\mathbb{N}$-grading defined by $\deg(\hbar) = \deg(a_p) = 1$. Since the map $\mathcal{D}^Y : Y_i^0 \to R(m)$ preserves the grading, it extends to a homomorphism $\widehat{Y_i^0} \to \widehat{R(m)}$.

Corollary *Let* $ch : S(m) \to R(m)$ *be the homomorphism defined by*

$$q \longmapsto e^{\hbar/2} \quad \textit{and} \quad A_p \longmapsto e^{a_p}$$

Then, the following diagram commutes

$$\begin{array}{ccc} \widetilde{U}_i^0 & \xrightarrow{\mathcal{D}^U} & S(m) \\ {\scriptstyle \Phi^0}\downarrow & & \downarrow{\scriptstyle ch} \\ \widehat{Y_i^0} & \xrightarrow{\mathcal{D}^Y} & \widehat{R(m)} \end{array}$$

where Φ^0 *is defined in Sect. 3.1.*

Proof It suffices to check the commutativity on the generators $\{H_{i,r}\}_{r\in\mathbb{Z}}$ of $\widetilde{U}_i^0$. The statement now follows from (4.7), (4.10) and the fact that, for $r \neq 0$, $\Phi^0(H_{i,r}) = B_i(v)/(q_i - q_i^{-1})\Big|_{v=r}$. ◻

4.6 Proof of Proposition 4.2

By Corollaries 4.4 and 4.5, it suffices to prove that, for any $k \in \mathbb{Z}$,

$$\mathrm{ch}\left(\mathcal{D}^U\left(\frac{\psi_{i,k} - \phi_{i,k}}{q_i - q_i^{-1}}\right)\right) = \frac{\hbar}{q_i - q_i^{-1}}\mathcal{D}^Y\left(\mathrm{e}^{kv}\exp\left(\gamma_i(v)\right)\Big|_{v^n=\xi_n}\right)$$

By (4.5)–(4.6) and (4.12), the left-hand side is equal to

$$\mathrm{ch}\left(\sum_p A_p^k \prod_{p'\neq p} \frac{q_i A_p - q_i^{-1} A_{p'}}{A_p - A_{p'}}\right)$$

We now compute the right-hand side. By (4.2) and (4.10),

$$\mathcal{D}^Y(\gamma_i(v)) = \sum_p e^{-a_p\partial_v}\frac{1-e^{d_i\hbar\partial_v}}{\partial_v}\partial_v G(v) = \sum_p G(v-a_p) - G(v-a_p+d_i\hbar)$$

where we used $e^{a\partial_v}G(v) = G(v+a)$. Thus,

$$\mathcal{D}^Y\left(e^{kv}\exp(\gamma_i(v))\right) = e^{kv}\prod_p \frac{v-a_p}{v-a_p+d_i\hbar}\frac{q_i e^v - q_i^{-1}e^{a_p}}{e^v - e^{a_p}}$$

By (4.8), the substitution $v^n = \xi_{i,n}$ in a formal power series $F \in \widehat{Y_i^0[v]}$ gives

$$\mathcal{D}^Y(F(v)|_{v^n=\xi_n}) = d_i\sum_p \mathcal{D}^Y(F(a_p))\prod_{p'\neq p}\frac{a_p - a_{p'} + d_i\hbar}{a_p - a_{p'}}$$

Since $(v-a_p)/(e^v - e^{a_p})\big|_{v=a_p} = e^{-a_p}$ and $(q_i e^v - q_i^{-1}e^{a_p})/(v-a_p+d_i\hbar)\Big|_{v=a_p} = e^{a_p}(q_i - q_i^{-1})/d_i\hbar$, this implies that

$$\begin{aligned}
&\mathcal{D}^Y\left(e^{kv}\exp\left(\gamma_i(v)\right)|_{v^n=\xi_{i,n}}\right)\\
&= \frac{q_i - q_i^{-1}}{\hbar}\sum_p e^{ka_p}\prod_{p'\neq p}\frac{a_p - a_{p'}}{a_p - a_{p'} + d_i\hbar}\frac{q_i e^{a_p} - q_i^{-1}e^{a_{p'}}}{e^{a_p} - e^{a_{p'}}}\frac{a_p - a_{p'} + d_i\hbar}{a_p - a_{p'}}\\
&= \frac{q_i - q_i^{-1}}{\hbar}\sum_p e^{ka_p}\prod_{p'\neq p}\frac{q_i e^{a_p} - q_i^{-1}e^{a_{p'}}}{e^{a_p} - e^{a_{p'}}}
\end{aligned}$$

as claimed.

4.7 A joint solution to equations (A)–(C)

Proposition 4.2 suggests replacing equation (B) of Theorem 3.4 by the stronger requirement that, for any $i \in \mathbf{I}$

$$g_i^+(v)\lambda_i^+(v)(g_i^-(v)) = \frac{\hbar}{q_i - q_i^{-1}}\exp(\gamma_i(v)) \qquad (\widetilde{B})$$

Using equation (A) for $j = i$ and $u = v$ then shows that

$$g_i^-(v)\lambda_i^-(v)(g_i^+(v)) = \frac{\hbar}{q_i - q_i^{-1}}\exp(\gamma_i(v))$$

Applying the operator $\lambda_i^-(v)$ to the first of these equations, and using $\lambda_i^+(v)\lambda_i^-(v) = \mathrm{id}$ (Proposition 2.10) and the second equation, yields $\lambda_i^-(v)(\gamma_i(v)) = \gamma_i(v)$. Similarly, applying $\lambda_i^+(v)$ to the second of these equations yields $\lambda_i^+(v)(\gamma_i(v)) = \gamma_i(v)$. This suggests in turn that a solution to the above equations may be given by $g_i^\pm(v) = g_i(v)$, where

$$g_i(v) = \left(\frac{\hbar}{q_i - q_i^{-1}}\right)^{1/2} \exp\left(\frac{\gamma_i(v)}{2}\right) \tag{4.14}$$

We now show that this is indeed the case.

Theorem *The series $g_i^\pm(v) = g_i(v)$ satisfy the conditions (A),($\widetilde{B}$) and (C) of Theorem 3.4, and therefore give rise to a homomorphism $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar\mathfrak{g}}$.*

4.8

We shall need the following

Lemma *Let $i, j \in \mathbf{I}$, and set $a = d_i a_{ij}/2$. Then,*

$$\lambda_i^\pm(u)\left(g_j(v)\right) = g_j(v) \exp\left(\pm\frac{G(v-u+a\hbar) - G(v-u-a\hbar)}{2}\right)$$

where $G(v)$ is given by (4.1).

Proof By Proposition 2.10,

$$\lambda_i^\pm(u)(B_j(v)) = B_j(v) \mp \frac{e^{a\hbar v} - e^{-a\hbar v}}{v} e^{uv}$$

Since $\gamma_j(v) = -B_j(-\partial_v)\partial_v G(v)$, we get

$$\begin{aligned}\lambda_i^\pm(u)\gamma_j(v) &= \gamma_j(v) \pm \frac{e^{a\hbar\partial_v} - e^{-a\hbar\partial_v}}{\partial_v} e^{-u\partial_v}\partial_v G(v) \\ &= \gamma_j(v) \pm (G(v-u+a\hbar) - G(v-u-a\hbar))\end{aligned}$$

The claim follows by exponentiating. □

4.9 Proof of condition (A)

We need to prove that for every $i, j \in \mathbf{I}$, we have

$$g_i(u)\lambda_i^+(u)(g_j(v)) = g_j(v)\lambda_j^-(v)(g_i(u))$$

By Lemma 4.8, this is equivalent to

$$g_i(u)g_j(v)\exp\left(\frac{G(v-u+a\hbar)-G(v-u-a\hbar)}{2}\right)$$
$$=g_i(u)g_j(v)\exp\left(\frac{G(u-v-a\hbar)-G(u-v+a\hbar)}{2}\right)$$

The result now follows since G is an even function.

4.10 Proof of condition $(\widetilde{B})$

Lemma 4.8 implies that

$$\begin{aligned} g_i(u)\lambda_i^+(u)(g_i(u)) &= g_i(u)^2\exp\left(\frac{G(d_i\hbar)-G(-d_i\hbar)}{2}\right) \\ &= g_i(u)^2 \\ &= \frac{\hbar}{q_i-q_i^{-1}}\exp\left(\gamma_i(u)\right) \end{aligned}$$

where the second equality holds because G is even.

4.11 Proof of condition (C)

Let $i, j \in \mathbf{I}$ and set $a = d_i a_{ij}/2$. We need to prove that

$$g_i(u)\lambda_i^\pm(u)(g_j(v))\frac{e^u-e^{v\pm a\hbar}}{u-v\mp a\hbar}=g_j(v)\lambda_j^\pm(v)(g_i(u))\frac{e^v-e^{u\pm a\hbar}}{v-u\mp a\hbar}$$

By Lemma 4.8 and the fact that G is even, we get the following equivalent assertion

$$\exp\left(G(v-u\pm a\hbar)-G(v-u\mp a\hbar)\right)\frac{e^u-e^{v\pm a\hbar}}{u-v\mp a\hbar}=\frac{e^v-e^{u\pm a\hbar}}{v-u\mp a\hbar}$$

Using the definition of G, the above becomes the equality

$$\left(\frac{v-u\pm a\hbar}{e^{v\pm a\hbar}-e^u}\right)\left(\frac{e^v-e^{u\pm a\hbar}}{v-u\mp a\hbar}\right)\left(\frac{e^u-e^{v\pm a\hbar}}{u-v\mp a\hbar}\right)=\frac{e^v-e^{u\pm a\hbar}}{v-u\mp a\hbar}$$

5 Uniqueness of homomorphisms

The aim of this section is to prove that homomorphisms of geometric type are unique up to conjugation and scaling.

5.1

Let $\mathcal{G}$ be the set of solutions $\mathbf{g} = \{g_i^{\pm}(u)\}_{i\in\mathbf{I}}$ of equations (A)–(C) of Theorem 3.4. Given a collection $\mathbf{r} = \{r_i^{\pm}(u)\}_{i\in\mathbf{I}}$ of invertible elements of $\widehat{Y^0[u]}$, set

$$\mathbf{r}\cdot\mathbf{g} = \{r_i^{\pm}(u)\cdot g_i^{\pm}(u)\}_{i\in\mathbf{I}}$$

Lemma *Let* $\mathbf{g}\in\mathcal{G}$*. Then,* $\mathbf{r}\cdot\mathbf{g}\in\mathcal{G}$ *if and only if the following holds*

(A$_0$) For any $i, j\in\mathbf{I}$*,*

$$r_i^+(u)\lambda_i^+(u)(r_j^-(v)) = r_j^-(v)\lambda_j^-(v)(r_i^+(u))$$

(B$_0$) For any $i\in\mathbf{I}$*,*

$$r_i^+(u)\lambda_i^+(u)(r_i^-(u)) = 1 = r_i^-(u)\lambda_i^-(u)(r_i^+(u))$$

(C$_0^{\pm}$) For any $i, j\in\mathbf{I}$*,*

$$r_i^{\pm}(u)\lambda_i^{\pm}(u)(r_j^{\pm}(v)) = r_j^{\pm}(v)\lambda_j^{\pm}(v)(r_i^{\pm}(u))$$

Proof Let $\mathbf{h} = \mathbf{r}\cdot\mathbf{g}$. The following assertions are straightforward to check

- $\mathbf{h}$ satisfies (A) if and only if $\mathbf{r}$ satisfies (A$_0$).
- $\mathbf{h}$ satisfies (B) if $\mathbf{r}$ satisfies (B$_0$).
- $\mathbf{h}$ satisfies (C) if and only if $\mathbf{r}$ satisfies (C$_0^{\pm}$).

There remains to prove that if $\mathbf{h}$ lies in $\mathcal{G}$, then $\mathbf{r}$ satisfies (B$_0$).

We claim that (A$_0$) and (C$_0^{\pm}$) imply that $c_i(u) = r_i^+(u)\lambda_i^+(u)(r_i^-(u))$ lies in $\mathbb{C}[[\hbar, u]]$. Assuming this, write $c_i(u) = \sum_n c_i^{(n)}u^n$, where $c_i^{(n)}\in\mathbb{C}[[\hbar]]$. Then,

$$\begin{aligned}\left(h_i^+(u)\lambda_i^+(u)(h_i^-(u))\right)\big|_{u^m=\xi_{i,m}} &= \left(c_i(u)g_i^+(u)\lambda_i^+(u)(g_i^-(u))\right)\big|_{u^m=\xi_{i,m}}\\ &= \sum_{n\geq 0} c_i^{(n)}\left(g_i^+(u)\lambda_i^+(u)(g_i^-(u))\right)\big|_{u^m=\xi_{i,m+n}}\\ &= c_i(\sigma_i^0)\left(g_i^+(u)\lambda_i^+(u)(g_i^-(u))\right)\big|_{u^m=\xi_{i,m}}\end{aligned}$$

where $\sigma_i^0 : Y^0\to Y^0$ is the algebra homomorphism defined by $\sigma_i^0(\xi_{j,m}) = \xi_{j,m+\delta_{ij}}$. Since both $\mathbf{h}$ and $\mathbf{g}$ satisfy (B) with $k=0$, this yields

$$\Phi^0\left(\frac{e^{\frac{\hbar d_i}{2}H_{i,0}} - e^{-\frac{\hbar d_i}{2}H_{i,0}}}{q_i - q_i^{-1}}\right) = c_i(\sigma_i^0)\Phi^0\left(\frac{e^{\frac{\hbar d_i}{2}H_{i,0}} - e^{-\frac{\hbar d_i}{2}H_{i,0}}}{q_i - q_i^{-1}}\right)$$

An inductive argument using the $\mathbb{C}[\hbar]$-linear $\mathbb{N}$-grading on Y^0 given by $\deg(\xi_{j,m}) = m$ and $\deg(\hbar) = 0$ then shows that $c_i^{(0)} = 1$ and $c_i^{(n)} = 0$ for any $n\geq 1$, so that $\mathbf{r}$ satisfies (B$_0$).

To prove our claim, set

$$c_i(u) = r_i^+(u)\lambda_i^+(u)(r_i^-(u)) = r_i^-(u)\lambda_i^-(u)(r_i^+(u))$$

so that $r_i^-(u) = c_i(u)\lambda_i^-(u)(r_i^+(u))^{-1}$. By (A_0), the following holds for every $i, j \in \mathbf{I}$

$$r_i^+(u)\lambda_i^+(u)\left(c_j(v)\lambda_j^-(v)(r_j^+(v))^{-1}\right) = c_j(v)\lambda_j^-(v)(r_j^+(v)^{-1})\lambda_j^-(v)(r_i^+(u))$$

Since $\lambda_i^+(u)$ and $\lambda_j^-(v)$ commute, we get

$$\begin{aligned}\lambda_i^+(u)(c_j(v))\left(r_i^+(u)\lambda_j^-(v)(r_j^+(v))\right) &= c_j(v)\lambda_j^-(v)\left(r_i^+(u)\lambda_i^+(u)(r_j^+(v))\right)\\ &= c_j(v)\lambda_j^-(v)\left(r_j^+(v)\lambda_j^+(v)(r_i^+(u))\right)\\ &= c_j(v)\left(r_i^+(u)\lambda_j^-(v)(r_j^+(v))\right)\end{aligned}$$

where the second equality uses (C_0^+) and the third one $\lambda_j^-(v)\lambda_j^+(v) = 1$. We have therefore proved that

$$\lambda_i^+(u)(c_j(v)) = c_j(v) \quad \text{for every } i, j \in \mathbf{I}$$

By definition of the operators $\lambda_i^\pm$, this implies that the coefficients of $c_j(v)$ lie in the centre of $Y_\hbar(\mathfrak{g})$, which is trivial. □

5.2

The uniqueness of homomorphisms of geometric type relies on the following

Proposition *Let* $\{r_i^+(u)\}_{i\in\mathbf{I}} \subset 1 + \widehat{Y^0[u]}_+$ *be a collection of invertible elements satisfying condition* (C_0^+) *of Lemma 5.1. Then, there exists an element* $\xi \in 1 + \widehat{Y^0}_+$ *such that, for any* $i \in \mathbf{I}$

$$r_i^+(u) = \xi \cdot \lambda_i^+(u)(\xi)^{-1}$$

Moreover, if $\zeta \in \widehat{Y^0}^\times$ *is any element such that* $r_i^+(u) = \zeta \cdot \lambda_i^+(u)(\zeta)^{-1}$*, then* $\zeta = c\xi$ *for some* $c \in \mathbb{C}[[\hbar]]^\times$.

The proof of Proposition 5.2 is given in Sects. 5.3–5.9.

5.3

We begin by linearising the problem. Set

$$\overline{r}_i(u) = \log(r_i(u)) \in \widehat{Y^0[u]}_+$$

By condition (C_0^+), the following holds for any $i, j \in \mathbf{I}$

$$(\lambda_i^+(u) - 1)(\overline{r}_j(v)) = (\lambda_j^+(v) - 1)(\overline{r}_i(u)) \tag{5.1}$$

and we need to show that

$$\overline{r}_i(u) = (\lambda_i^+(u) - 1)\eta \tag{5.2}$$

for some $\eta \in \widehat{Y^0}_+$.

5.4 Rank 1 case

We assume first that $|\mathbf{I}| = 1$ and accordingly drop the subscript i from our formulae. We shall prove (5.2) by working with an adapted system of generators of Y^0.

Recall that, by Proposition 2.10,

$$(\lambda^+(u) - 1)B(v) = -\frac{e^{\hbar v} - e^{-\hbar v}}{v} e^{uv}$$

Define $B'(v) = \sum_{k \geq 0} \frac{v^k}{k!} t'_k$ by equating the coefficients of v in

$$B'(v) = -\frac{v}{e^{\hbar v} - e^{-\hbar v}} B(v) = -\frac{\hbar v}{e^{\hbar v} - e^{-\hbar v}} \sum_{n \geq 0} \frac{v^n}{n!} t_n$$

The elements $\{t'_k\}_{k \in \mathbb{N}}$ give another system of generators of Y^0 which are homogeneous, with $\deg(t'_k) = k = \deg(t_k)$ for any $k \in \mathbb{N}$, and satisfy

$$\lambda^+(u)(t'_k) = t'_k + u^k \tag{5.3}$$

5.5

Since the operator $\lambda^+(u) : Y^0 \to Y^0[u]$ is homogeneous with respect to the $\mathbb{N}$-grading extending that on Y^0 by $\deg(u) = 1$, it suffices to prove (5.2) when $\overline{r}(u)$ is homogeneous of degree $n \in \mathbb{N}$. Moreover, since $\lambda^+(u)$ is $\mathbb{C}[\hbar]$-linear and the formulae (5.3) do not involve $\hbar$, we may further assume that the coefficients of $\overline{r}(u)$ lie in the $\mathbb{C}$-subalgebra $\overline{Y^0} \subset Y^0$ generated by the $\{t'_k\}$.

An element of $\overline{Y^0}[u]_n$ has the form

$$\bar{r}(u) = \sum_{|\mu| \leq n} a_\mu t'_\mu u^{n-|\mu|} \tag{5.4}$$

where $a_\mu \in \mathbb{C}[t'_0]$ and, for a partition μ of length l, we define $t'_\mu = t'_{\mu_1} \dots t'_{\mu_l}$. The proof of the existence of $\eta \in \overline{Y^0}_n$ such that $(\lambda^+(u) - 1)(\eta) = \overline{r}(u)$ proceeds in two steps:

(1) Show that, modulo elements of the form $(\lambda^+(u) - 1)(\eta)$,

$$\bar{r}(u) = \sum_{|\mu|<n} a_\mu t'_\mu u^{n-|\mu|} \tag{5.5}$$

where $a_\mu \in \mathbb{C}$ do not depend on t'_0.

(2) Show that any $\overline{r}(u)$ of the form (5.5) is equal to $(\lambda^+(u) - 1)(\eta)$ for some $\eta \in \overline{Y^0}_n$.

5.6 Proof of (1)

For $\bar{r}(u) \in \overline{Y^0}_n$ of the form (5.4), choose $b_\nu \in \mathbb{C}[t'_0]$ for every $\nu \vdash n$ such that

$$b_\nu(t'_0 + 1) - b_\nu(t'_0) = a_\nu(t'_0)$$

Then

$$\bar{r}(u) - (\lambda^+(u) - 1)\left(\sum_{\nu\vdash n} b_\nu t'_\nu\right) = \sum_{|\mu|<n} a'_\mu t'_\mu u^{n-|\mu|}$$

for some $a'_\mu \in \mathbb{C}[t'_0]$, so that we may assume that $\overline{r}(u)$ is of the form (5.4) with $a_\mu = 0$ for any $\mu \vdash n$.

Write now

$$\begin{aligned}(\lambda^+(v) - 1)\overline{r}(u) &= \sum_{|\mu|<n} \left(a_\mu(t_0 + 1) - a_\mu(t_0)\right) t'_\mu u^{n-|\mu|} \\ &\quad + \sum_{|\mu|<n} a_\mu(t_0 + 1) \left(\sum_{\nu \subsetneq \mu} c(\nu, \mu) t'_\nu v^{|\mu|-|\nu|}\right) u^{n-|\mu|}\end{aligned}$$

where $c(\nu, \mu)$ is the number of ways of obtaining ν by removing rows from μ. By (5.1), the above expression is symmetric in u and v. Its value at $u = 0$, which is 0, must therefore equal its value at $v = 0$, thus leading to

$$\sum_{|\mu|<n} \left(a_\mu(t_0 + 1) - a_\mu(t_0)\right) t'_\mu u^{n-|\mu|} = 0$$

which implies that $a_\mu \in \mathbb{C}$ for any μ.

5.7 Proof of (2)

Let $\bar{r}(u)$ be of the form (5.5). For any $0 \le l \le n$, write

$$\bar{r}_l(u) = \sum_{\substack{|\mu|<n \\ l(\mu)=l}} a_\mu t'_\mu u^{n-|\mu|}$$

so that $\bar{r}(u) = \sum_l \bar{r}_l(u)$. We proceed by induction on the largest positive integer k such that $\bar{r}_k(u) \neq 0$. If $k = 0$, then $\bar{r}(u) = cu^n = (\lambda^+(u) - 1)(ct'_n)$.

Assume now that $k > 0$ and let $D(u) : \overline{Y^0} \to \overline{Y^0}[u]$ be the differential operator $D(u) = \sum_{m\ge 1} u^m \partial_{t'_m}$. Since $(\lambda^+(u) - 1)(t'_k) = u^k$ we get, for any partition μ

$$(\lambda^+(u) - 1)(t'_\mu) = D(u)(t'_\mu) + \text{terms of smaller length}$$

Thus, (5.1) implies that

$$D(u)\bar{r}_k(v) = D(v)\bar{r}_k(u)$$

This cross-derivative condition implies the existence of $\eta \in \overline{Y^0}_n$ such that $r_k(u) = D(u)\eta$. This implies that $\bar{r}(u) - (\lambda^+(u) - 1)(\eta)$ has smaller k.

This completes the proof of the existence part of Proposition 5.2 when $\mathfrak{g}$ is of rank 1.

5.8 Arbitrary rank

The argument for arbitrary $\mathfrak{g}$ rests on the following

Lemma *There exist generators* $\{\varpi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ *of* Y^0 *which are homogeneous, with* $\deg(\varpi_{i,r}) = r$ *and such that*

$$\lambda_i^\pm(u)\varpi_{j,r} = \varpi_{j,r} \pm \delta_{i,j} u^r$$

Proof By Proposition 2.10, the generating series $B_j(v) = \hbar \sum_{r\ge 0} t_{j,r} v^r/r!$ satisfy

$$(\lambda_i^\pm(u) - 1)\hbar^{-1} B_j(v) = \mp Q_{ij}(v) e^{uv}$$

where $Q_{ij}(v) = 2\sinh(\hbar d_i a_{ij} v/2)/\hbar v$. Since $Q_{ij} = d_i a_{ij} \mod \hbar$, the matrix $Q = (Q_{ij})$ is invertible. Set $B'_i(v) = -\hbar^{-1} \sum_j Q_{ij}^{-1} B_j(v)$. Then $(\lambda_i^\pm(u) - 1)B'_j(v) = \pm\delta_{ij} e^{uv}$ which, in terms of the expansion $B'_i(v) = \sum \varpi_{i,r} v^r/r!$ yields the required transformation property.

Since $\deg(v) = 1$, the stated homogeneity of the $\varpi_{i,r}$ is equivalent to $\mathrm{Ad}(\zeta)(B'_i(v)) = B'_i(\zeta^2 v)$ where $\mathrm{Ad}(\zeta)$ denotes the action of $\zeta \in \mathbb{C}^\times$ on $Y_\hbar(\mathfrak{g})[[v]]$ corresponding to the $\mathbb{N}$-grading. This in turn follows from the fact that $\mathrm{Ad}(\zeta)(\hbar^{-1} B_j(v)) = B_j(\zeta^2 v)$ and $\mathrm{Ad}(\zeta) Q(v) = Q(\zeta^2 v)$. ◻

Using the generators $\varpi_{i,r}$, the proof of the existence part of Proposition 5.2 in higher rank follows the same argument as the one used for proving the sufficiency of the cross-derivative condition (here the existence of a primitive for any $i \in \mathbf{I}$ is guaranteed by the rank 1 case).

5.9 Uniqueness of ξ

Let $\zeta \in \widehat{Y^0}^\times$ be an element such that $r_i^+(u) = \zeta \cdot \lambda_i^+(u)(\zeta)^{-1}$ for each $i \in \mathbf{I}$. Then

$$\begin{aligned}\lambda_i^+(u)(\zeta\xi^{-1}) &= \lambda_i^+(u)(\zeta)\lambda_i^+(u)(\xi)^{-1}\\ &= r_i^+(u)^{-1}\zeta r_i^+(u)\xi^{-1}\\ &= \zeta\xi^{-1}\end{aligned}$$

By Proposition 2.10 (6), we get that $\lambda_i^\pm(u)(\zeta\xi^{-1}) = \zeta\xi^{-1}$ for each $i \in \mathbf{I}$. By definition of the operators $\lambda_i^\pm(u)$, this implies that the coefficients of $\zeta\xi^{-1}$ lie in the centre of $Y_\hbar(\mathfrak{g})$, which is trivial. This completes the proof of the last assertion of Proposition 5.2.

5.10 Torus action

The adjoint action of $\mathfrak{h}$ on $Y_\hbar(\mathfrak{g})$ exponentiates to one of the algebraic torus $H = \mathrm{Hom}_\mathbb{Z}(Q, \mathbb{C}^\times)$ where $Q \subset \mathfrak{h}^*$ is the root lattice. This action preserves homomorphisms of geometric type and acts on the corresponding formal power series by $\zeta \cdot \{g_i^\pm(u)\} = \{\zeta_i^{\pm 1} g_i^\pm(u)\}$ where $H \ni \zeta \to \zeta_i = \zeta(\alpha_i)$ is the ith coordinate function on H.

5.11 Uniqueness of homomorphisms of geometric type

Theorem *Let $\Phi, \Phi' : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ be two homomorphisms of geometric type. Then, there exists $\zeta \in H$ and $\xi \in 1 + \widehat{Y^0}_+$ such that*

$$\Phi' = Ad(\xi) \circ (\zeta \cdot \Phi)$$

Moreover, ζ is unique and ξ is unique up to multiplication by $c \in \mathbb{C}[[\hbar]]^\times$.

Proof Let $\{g_i^\pm(u)\}, \{h_i^\pm(u)\} \subset \widehat{Y^0[[u]]}$ be elements of $\mathcal{G}$ corresponding to Φ and Φ' respectively. By Lemma 3.7, we may use the action of H to assume that $g_i^\pm(u) = h_i^\pm(u)$ mod $\widehat{Y^0[u]}_+$. By Lemma 5.1, the elements $r_i^\pm(u) = h_i^\pm(u) \cdot g_i^\pm(u)^{-1} \in 1 + \widehat{Y^0[u]}_+$ satisfy conditions (A_0)–($\mathrm{C}_0^\pm$). By Proposition 5.2, we may find an element $\xi \in 1 + \widehat{Y^0}_+$ such that $r_i^+(u) = \xi \cdot \lambda_i^+(u)(\xi^{-1})$. It follows that for any $i \in \mathbf{I}$

$$\begin{aligned}\Phi'(E_{i,0}) &= h_i^+(\sigma_i^+)x_{i,0}^+\\ &= r_i^+(\sigma_i^+)g_i^+(\sigma_i^+)x_{i,0}^+\end{aligned}$$

$$= \xi \lambda_i^+(\sigma_i^+)(\xi^{-1}) g_i^+(\sigma_i^+) x_{i,0}^+$$
$$= \xi g_i^+(\sigma_i^+) x_{i,0}^+ \xi^{-1}$$

Moreover, for any $r \in \mathbb{Z}$,

$$\Phi'(E_{i,r}) = \mathrm{e}^{r\sigma_i^+} \Phi'(E_{i,0}) = \mathrm{e}^{r\sigma_i^+} \xi \Phi(E_{i,0}) \xi^{-1} = \xi \Phi(E_{i,r}) \xi^{-1}$$

By (B_0), $r_i^-(u) = \lambda_i^-(u)(r_i^+(u)^{-1}) = \xi \lambda_i^-(u)(\xi^{-1})$ and it follows similarly that $\Phi'(F_{i,r}) = \xi \Phi(F_{i,r}) \xi^{-1}$ for any $i \in \mathbf{I}$ and $r \in \mathbb{Z}$. Since Φ and Φ' coincide on U^0 and $\mathrm{Ad}(\xi)(\eta) = \eta$ for any $\eta \in Y^0$ it follows that $\Phi' = \mathrm{Ad}(\xi) \circ \Phi$. The last assertion of Proposition 5.2 implies the uniqueness of ξ up to multiplication by an element of $\mathbb{C}[[\hbar]]^\times$. □

6 Isomorphisms of geometric type

We prove in this section that any homomorphism of geometric type $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ extends to an isomorphism of completed algebras and induces Drinfeld's degeneration of $U_\hbar(L\mathfrak{g})$ to $Y_\hbar(\mathfrak{g})$.

6.1 Classical limit

The specialisations of the quantum loop algebra $U_\hbar(L\mathfrak{g})$ and Yangian $Y_\hbar(\mathfrak{g})$ at $\hbar = 0$ are the enveloping algebras $U(\mathfrak{g}[z, z^{-1}])$ and $U(\mathfrak{g}[s])$ respectively. Specifically, if $\{e_i, f_i, h_i\}_{i \in \mathbf{I}}$ are the generators of $\mathfrak{g}$ given in Sect. 2.1, the assignments

$$e_i \otimes z^k \to E_{i,k}, \qquad f_i \otimes z^k \to F_{i,k}, \qquad h_i \otimes z^r \to H_{i,r}$$

and

$$e_i \otimes s^r \to \frac{1}{\sqrt{d_i}} x_{i,r}^+, \qquad f_i \otimes s^r \to \frac{1}{\sqrt{d_i}} x_{i,r}^-, \qquad h_i \otimes s^r \to \frac{1}{d_i} \xi_{i,r}$$

extend respectively to isomorphisms

$$U(\mathfrak{g}[z, z^{-1}]) \xrightarrow{\sim} U_\hbar(L\mathfrak{g})/\hbar U_\hbar(L\mathfrak{g}) \qquad \text{and} \qquad U(\mathfrak{g}[s]) \xrightarrow{\sim} Y_\hbar(\mathfrak{g})/\hbar Y_\hbar(\mathfrak{g})$$

Proposition *Let $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar(\mathfrak{g})}$ be the homomorphism given by Theorem 4.7. Then, the specialisation of Φ at $\hbar = 0$ is the homomorphism*

$$\exp^* : U(\mathfrak{g}[z, z^{-1}]) \longrightarrow U(\mathfrak{g}[[s]]) \subset \widehat{U(\mathfrak{g}[s])}$$

given on $\mathfrak{g}[z, z^{-1}]$ by $\exp^*(X \otimes z^k) = X \otimes e^{ks}$.

Proof Since $\Phi(H_{i,0}) = d_i^{-1} t_{i,0}$ and, for $r \neq 0$,

$$\Phi(H_{i,r}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k \geq 0} t_{i,k} \frac{r^k}{k!}$$

setting $\hbar = 0$ yields $\Phi|_{\hbar=0}(h_i \otimes z^0) = h_i \otimes s^0$, and

$$\Phi|_{\hbar=0}(h_i \otimes z^r) = \frac{1}{d_i} \sum_{k \geq 0} d_i h_i \otimes \frac{s^k r^k}{k!} = h_i \otimes \mathrm{e}^{rs}$$

Further, since $g_i^+(u) = \frac{1}{\sqrt{d_i}} \mod \hbar$ by (4.14), we get

$$\Phi|_{\hbar=0}(e_i \otimes z^r) = \frac{1}{\sqrt{d_i}} \mathrm{e}^{r\sigma_i^+} \sqrt{d_i} e_i \otimes s^0 = \sum_{k \geq 0} e_i \otimes \frac{s^k r^k}{k!} = e_i \otimes \mathrm{e}^{rs}$$

where we used the fact that, in the classical limit, the operator σ_i^+ corresponds to multiplication by s. Similarly, $\Phi|_{\hbar=0}(f_i \otimes z^r) = f_i \otimes \mathrm{e}^{rs}$. □

6.2

Let $\mathcal{J} \subset U_\hbar L\mathfrak{g}$ be the kernel of the composition

$$U_\hbar L\mathfrak{g} \xrightarrow{\hbar \to 0} U(L\mathfrak{g}) \xrightarrow{z \to 1} U\mathfrak{g}$$

and let

$$\widehat{U_\hbar(L\mathfrak{g})} = \varprojlim U_\hbar(L\mathfrak{g})/\mathcal{J}^n$$

be the completion of $U_\hbar(L\mathfrak{g})$ with respect to the ideal $\mathcal{J}$.

Theorem *Let $\Phi : U_\hbar(L\mathfrak{g}) \to \widehat{Y_\hbar \mathfrak{g}}$ be a homomorphism of geometric type. Then,*

(1) *Φ maps $\mathcal{J}$ to the ideal $\widehat{Y_\hbar(\mathfrak{g})}_+ = \prod_{n \geq 1} Y_\hbar(\mathfrak{g})_n$.*
(2) *The corresponding homomorphism*

$$\widehat{\Phi} : \widehat{U_\hbar(L\mathfrak{g})} \to \widehat{Y_\hbar(\mathfrak{g})}$$

is an isomorphism.

Proof (1) Note first that $\mathcal{J}$ is generated by $\hbar U_\hbar(L\mathfrak{g})$ and the elements $\{H_{i,r} - H_{i,s}, E_{i,r} - E_{i,s}, F_{i,r} - F_{i,s}\}_{i \in \mathbf{I}, r,s \in \mathbb{Z}}$ since its image in $U(\mathfrak{g}[z, z^{-1}])$ is generated by the classes of these elements. Note next that, for $r, s \neq 0$

$$\Phi(H_{i,r} - H_{i,s}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k\geq 1} \frac{r^k - s^k}{k!} t_{i,k}$$

while

$$\Phi(H_{i,r} - H_{i,0}) = \frac{\hbar}{q_i - q_i^{-1}} \sum_{k\geq 1} \frac{r^k}{k!} t_{i,k} + \left(\frac{\hbar}{q_i - q_i^{-1}} - d_i^{-1}\right) t_{i,0}$$

which lies in $\prod_{n\geq 1} Y_\hbar(\mathfrak{g})_n$ since $\hbar/(q_i - q_i^{-1}) = d_i^{-1} \mod \hbar$. Finally, for $r, s \in \mathbb{Z}$,

$$\Phi(E_{i,r} - E_{i,s}) = (\mathrm{e}^{r\sigma_i^+} - \mathrm{e}^{s\sigma_i^+}) g_i^+(\sigma_i^+) e_{i,0} \in \mathcal{J}$$

and similarly $\Phi(F_{i,r} - F_{i,s}) \in \mathcal{J}$.

(2) By Theorem 5.11, it suffices to prove this for the explicit homomorphism given by Theorem 4.7. The result then follows Proposition 6.3 below and the fact that, by Proposition 6.1, the specialisation of $\widehat{\Phi}$ at $\hbar = 0$ is an isomorphism $\widehat{U(\mathfrak{g}[z, z^{-1}])} \to \widehat{U(\mathfrak{g}[s])}$. □

6.3

Let $J \subset U(\mathfrak{g}[z, z^{-1}])$ be the kernel of evaluation at $z = 1$ and $\widehat{U(L\mathfrak{g})}$ the completion of $U(L\mathfrak{g})$ with respect to J.

Proposition (1) *$\widehat{U_\hbar(L\mathfrak{g})}$ is a flat deformation of $\widehat{U(\mathfrak{g}[z, z^{-1}])}$.*
(2) *$\widehat{Y_\hbar(\mathfrak{g})}$ is a flat deformation of $\widehat{U(\mathfrak{g}[s])}$ over $\mathbb{C}[[\hbar]]$.*

Proof (1) Set, for brevity $\mathcal{U} = U_\hbar(L\mathfrak{g})$ and $U = U(\mathfrak{g}[z, z^{-1}])$. We claim that $\widehat{\mathcal{U}}$ is a flat deformation of $\widehat{\mathcal{U}}/\hbar\widehat{\mathcal{U}}$, and that $\widehat{\mathcal{U}}/\hbar\widehat{\mathcal{U}}$cong$\widehat{U}$.

To prove the first assertion it suffices to show, by [21, Prop XVI.2.4], that $\widehat{\mathcal{U}}$ is a separated, complete and torsion-free $\mathbb{C}[[\hbar]]$-module. To show that it is separated, note that $\hbar \in \mathcal{J}$, so that $\hbar^k\widehat{\mathcal{U}} \subset \lim\limits_{\substack{\longleftarrow \\ n>k}} \mathcal{J}^k/\mathcal{J}^n$ and

$$\bigcap_{k\geq 0} \hbar^k\widehat{\mathcal{U}} = \{0\}$$

To show completeness, note that

$$\widehat{\mathcal{U}}/\hbar^k\widehat{\mathcal{U}} = \lim_{\substack{\longleftarrow \\ n}} (\mathcal{U}/\mathcal{J}^n)/(\hbar^k\mathcal{U}/\hbar^k\mathcal{U} \cap \mathcal{J}^n) = \lim_{\substack{\longleftarrow \\ n}} \begin{cases} \mathcal{U}/\mathcal{J}^n & \text{if } n \leq k \\ \mathcal{U}/\hbar^k\mathcal{U} + \mathcal{J}^n & \text{if } n > k \end{cases}$$

from which it readily follows that the map

$$\widehat{\mathcal{U}} \longrightarrow \lim_{\substack{\longleftarrow \\ k}} \widehat{\mathcal{U}}/\hbar^k\widehat{\mathcal{U}}$$

is surjective. Finally, to prove that $\widehat{\mathcal{U}}$ is torsion-free, note that the kernel of multiplication by $\hbar$ on $\mathcal{U}/\mathcal{J}^n$ is $\hbar^{-1}(\hbar\mathcal{U}\cap\mathcal{J}^n)/\mathcal{J}^n$. We claim that $\hbar\mathcal{U}\cap\mathcal{J}^n=\hbar\mathcal{J}^{n-1}$, which implies that the kernel of $\hbar$ on $\widehat{\mathcal{U}}$ is $\lim_n \mathcal{J}^{n-1}/\mathcal{J}^n=\{0\}$. To prove the claim, use the flatness of $\mathcal{U}$ to identify it with the $\mathbb{C}[[\hbar]]$-module $U[[\hbar]]$, so that $\mathcal{J}=J\oplus\hbar U[[\hbar]]$. Let $a_1,\ldots,a_n\in\mathcal{J}$ and write $a_i=a_i^0+\hbar\overline{a}_i$, where $a_i^0\in J$ and $\overline{a}_i\in U[[\hbar]]$. Then

$$a_1\ldots a_n=a_1^0a_2\ldots a_n \mod \hbar\mathcal{J}^{n-1}=\cdots=a_1^0\ldots a_n^0 \mod \hbar\mathcal{J}^{n-1}$$

from which the claim follows.

The fact that $\widehat{\mathcal{U}}/\hbar\widehat{\mathcal{U}}\cong\widehat{U}$ follows by taking limits in the sequence

$$0\to\hbar\left(\mathcal{U}/\mathcal{J}^{n-1}\right)\to\mathcal{U}/\mathcal{J}^n\to U/J^n\to 0$$

The latter is exact since, under the natural surjection $\mathcal{U}\to U$, the ideal $\mathcal{J}^n$ is mapped to J^n with kernel $\hbar\mathcal{U}\cap\mathcal{J}^n=\hbar\mathcal{J}^{n-1}$.

(2) Since $\widehat{Y_\hbar(\mathfrak{g})}$ is the completion of $Y_\hbar(\mathfrak{g})$ with respect to the ideal $Y_\hbar(\mathfrak{g})_+$ of elements of positive degree, it follows as in (1) that it is a separated and complete $\mathbb{C}[[\hbar]]$-module. The lack of torsion of $Y_\hbar(\mathfrak{g})$ implies that $\hbar Y_\hbar(\mathfrak{g})\cap Y_\hbar(\mathfrak{g})_+^n=\hbar Y_\hbar(\mathfrak{g})_+^{n-1}$ and therefore that $\widehat{Y_\hbar(\mathfrak{g})}$ is torsion-free. Thus, $\widehat{Y_\hbar(\mathfrak{g})}$ is a flat deformation of

$$\widehat{Y_\hbar(\mathfrak{g})}/\hbar\widehat{Y_\hbar(\mathfrak{g})}\cong Y_\hbar\widehat{(\mathfrak{g})/\hbar Y_\hbar}(\mathfrak{g})\cong\widehat{U(\mathfrak{g}[s])}$$

as claimed. □

6.4 Drinfeld's degeneration

Consider the descending filtration

$$U_\hbar(L\mathfrak{g})=\mathcal{J}^0\supset\mathcal{J}\supset\mathcal{J}^2\supset\cdots \tag{6.1}$$

defined by the powers of $\mathcal{J}$ and let $\mathrm{gr}_{\mathcal{J}}(U_\hbar(L\mathfrak{g}))=\bigoplus_{n\geq 0}\mathcal{J}^n/\mathcal{J}^{n+1}$ be its associated graded.

Theorem ([10,17]) *Let $\{d_i^\pm\}_{i\in\mathbf{I}}\subset\mathbb{C}^\times$ be such that $d_i^+d_i^-=d_i$. Then, the following assignment extends uniquely to an isomorphism of graded algebras $Y_\hbar(\mathfrak{g})\xrightarrow{\sim} gr_{\mathcal{J}}(U_\hbar(L\mathfrak{g}))$*

$$\xi_{i,0}\longmapsto d_iH_{i,0}\in U_\hbar(L\mathfrak{g})/\mathcal{J}$$
$$x_{i,0}^+\longmapsto d_i^+E_{i,0}\in U_\hbar(L\mathfrak{g})/\mathcal{J},\qquad x_{i,0}^-\longmapsto d_i^-F_{i,0}\in U_\hbar(L\mathfrak{g})/\mathcal{J}$$
$$x_{i,1}^+\longmapsto d_i^+(E_{i,1}-E_{i,0})\in\mathcal{J}/\mathcal{J}^2,\qquad x_{i,1}^-\longmapsto d_i^-(F_{i,1}-F_{i,0})\in\mathcal{J}/\mathcal{J}^2$$

Remark The fact that $U_\hbar(L\mathfrak{g})$ degenerates to $Y_\hbar(\mathfrak{g})$ is stated, without proof, in [10, §6]. The formulae above and the proof that they define an isomorphism $Y_\hbar(\mathfrak{g})\cong\mathrm{gr}_{\mathcal{J}}(U_\hbar\mathfrak{g})$ are given in [17].

6.5 Relation to Drinfeld's degeneration

By Theorem 6.2, a homomorphism of geometric type Φ induces a homomorphism

$$\mathrm{gr}(\Phi) : \mathrm{gr}_{\mathcal{J}}(U_\hbar(L\mathfrak{g})) \longrightarrow Y_\hbar(\mathfrak{g}) = \mathrm{gr}_{\widehat{Y_\hbar(\mathfrak{g})}_+} \widehat{Y_\hbar(\mathfrak{g})}$$

Let $\{g_i^\pm(v)\} \subset \widehat{Y^0[v]}^\times$ be the elements defining Φ. By Lemma 3.7,

$$g_i^\pm(v) = \frac{1}{d_i^\pm} \mod \widehat{Y^0[v]}_+ \tag{6.2}$$

for some $d_i^\pm \in \mathbb{C}^\times$ such that $d_i^+ d_i^- = d_i$.

Proposition *$gr(\Phi)$ is the inverse of the degeneration isomorphism $\imath : Y_\hbar(\mathfrak{g}) \xrightarrow{\sim} U_\hbar(L\mathfrak{g})$ given by Theorem 6.4.*

Proof It suffices to verify the claim on the generators $\{\xi_{i,0}, x_{i,0}^\pm, x_{i,1}^\pm\}_{i\in\mathbf{I}}$ of $Y_\hbar(\mathfrak{g})$. Now,

$$\mathrm{gr}(\Phi) \circ \imath(\xi_{i,0}) = \mathrm{gr}(\Phi)(d_i H_{i,0}) = \xi_{i,0}$$

and

$$\begin{aligned} \mathrm{gr}(\Phi) \circ \imath(x_{i,0}^+) &= d_i^+ \, \Phi(E_{i,0}) \mod \widehat{Y_\hbar(\mathfrak{g})}_+ \\ &= d_i^+ \, g_i^+(\sigma_i^+) x_{i,0}^+ \mod \widehat{Y_\hbar(\mathfrak{g})}_+ \\ &= x_{i,0}^+ \end{aligned}$$

by (6.2). Moreover,

$$\begin{aligned} \mathrm{gr}(\Phi) \circ \imath(x_{i,1}^+) &= d_i^+ \, \Phi(E_{i,1} - E_{i,0}) \mod \widehat{Y_\hbar(\mathfrak{g})}_{\geq 2} \\ &= d_i^+ (\mathrm{e}^{\sigma_i^+} - 1) g_i^+(\sigma_i^+) x_{i,0}^+ \mod \widehat{Y_\hbar(\mathfrak{g})}_{\geq 2} \\ &= x_{i,1}^+ \end{aligned}$$

And similarly $\mathrm{gr}(\Phi) \circ \imath(x_{i,r}^-) = x_{i,r}^-$ for $r = 0, 1$. □

7 Geometric solution for $\mathfrak{gl}_n$

In this section, we construct a homomorphism of geometric type for $\mathfrak{gl}_n$ and show that it intertwines the geometric realisations of the corresponding loop algebra and Yangian constructed by Ginzburg–Vasserot [15,34].

7.1 The quantum loop algebra [8]

Throughout this section, we fix $n \geq 2$ and mostly follow the notation of [34]. Set $\mathbf{I} = \{1, \ldots, n-1\}$ and $\mathbf{J} = \{1, \ldots, n\}$. Then, $U_\hbar(L\mathfrak{gl}_n)$ is topologically generated over $\mathbb{C}[[\hbar]]$ by elements $\{E_{i,r}, F_{i,r}, D_{j,r}\}_{i\in\mathbf{I}, j\in\mathbf{J}, r\in\mathbb{Z}}$. To describe the relations, introduce the formal power series

$$E_i(z) = \sum_{r\in\mathbb{Z}} E_{i,r} z^{-r} \qquad F_i(z) = \sum_{r\in\mathbb{Z}} F_{i,r} z^{-r}$$

and

$$\Theta_j^{\pm}(z) = \sum_{s\geq 0} \Theta^{\pm}_{j,\pm s} z^{\mp s} = \exp\left(\pm\frac{\hbar D_{j,0}}{2}\right) \exp\left(\pm(q-q^{-1})\sum_{s\geq 1} D_{j,\pm s} z^{\mp s}\right)$$

The relations are

(QL1-gl) For any $j, j' \in \mathbf{J}$ and $r, s \in \mathbb{Z}$,

$$[D_{j,r}, D_{j',s}] = 0$$

(QL2-gl) For any $i \in \mathbf{I}$ and $j \in \mathbf{J}$,

$$\Theta_j^{\pm}(z) E_i(w) \Theta_j^{\pm}(z)^{-1} = \vartheta_{c_{ji}}(q^{c_{ji}} z/w) E_i(w)$$
$$\Theta_j^{\pm}(z) F_i(w) \Theta_j^{\pm}(z)^{-1} = \vartheta_{c_{ji}}(q^{c_{ji}} z/w)^{-1} F_i(w)$$

where $c_{ji} = -\delta_{ji} + \delta_{j\,i+1}$, $\vartheta_m(\zeta) = \frac{q^m\zeta-1}{\zeta-q^m}$, and the right-hand side is expanded in powers of $z^{\mp 1}$.[2]

(QL3-gl) For any $i, i' \in \mathbf{I}$,

$$E_i(z) E_{i'}(w) = \vartheta_{a_{ii'}}(q^{i-i'} z/w) E_{i'}(w) E_i(z)$$
$$F_i(z) F_{i'}(w) = \vartheta_{a_{ii'}}(q^{i-i'} z/w)^{-1} F_{i'}(w) F_i(z)$$

where $a_{ii'} = 2\delta_{ii'} - \delta_{|i-i'|,1}$ are the entries of the Cartan matrix of $\mathfrak{sl}_n$ and the equalities are understood as holding after both side have been multiplied by the denominator of the function ϑ_m.

(QL4-gl) For any $i, i' \in \mathbf{I}$,

$$(q-q^{-1})[E_i(z), F_{i'}(w)] = \delta_{i,i'}\delta(z/w)\left(\frac{\Theta^+_{i+1}(z)}{\Theta^+_i(z)} - \frac{\Theta^-_{i+1}(z)}{\Theta^-_i(z)}\right)$$

where $\delta(\zeta) = \sum_{r\in\mathbb{Z}} \zeta^r$ is the formal delta function.

[2] note that the expansions in $z^{\pm 1}$ are related by the symmetry $\vartheta_m(\zeta^{-1}) = \vartheta_{-m}(\zeta)$.

(QL5-gl) For any $i, i' \in \mathbf{I}$ such that $|i - i'| = 1$,

$$E_i(z_1)E_i(z_2)E_{i'}(w) - (q+q^{-1})E_i(z_1)E_{i'}(w)E_i(z_2) + E_{i'}(w)E_i(z_1)E_i(z_2) + (z_1 \leftrightarrow z_2) = 0$$
$$F_i(z_1)F_i(z_2)F_{i'}(w) - (q+q^{-1})F_i(z_1)F_{i'}(w)F_i(z_2) + F_{i'}(w)F_i(z_1)F_i(z_2) + (z_1 \leftrightarrow z_2) = 0$$

For any $i, i' \in \mathbf{I}$ such that $|i - i'| \geq 2$,

$$E_i(z)E_{i'}(w) = E_{i'}(w)E_i(z)$$
$$F_i(z)F_{i'}(w) = F_{i'}(w)F_i(z)$$

In terms of the generators $\{E_{i,r}, F_{i,r}, D_{j,r}\}$, the relations (QL1-gl)–(QL4-gl) read

$$[D_{j,0}, E_{i,k}] = c_{ji}E_{i,k} \qquad [D_{j,r}, E_{i,k}] = q^{-c_{ji}r}\frac{[c_{ji}r]}{r}E_{i,k+r}$$
$$[D_{j,0}, F_{i,k}] = -c_{ji}F_{i,k} \qquad [D_{j,r}, F_{i,k}] = -q^{-c_{ji}r}\frac{[c_{ji}r]}{r}F_{i,k+r}$$
$$q^{i}E_{i,k+1}E_{i',l} - q^{a_{ii'}+i'}E_{i,k}E_{i',l+1} = q^{a_{ii'}+i}E_{i',l}E_{i,k+1} - q^{i'}E_{i',l+1}E_{i,k}$$
$$q^{a_{ii'}+i}F_{i,k+1}F_{i',l} - q^{i'}F_{i,k}F_{i',l+1} = q^{i}F_{i',l}F_{i,k+1} - q^{a_{ij}+i'}F_{i',l+1}F_{i,k}$$
$$[E_{i,k}, F_{i',l}] = \delta_{ii'}\frac{P^{+}_{i,k+l} - P^{-}_{i,k+l}}{q-q^{-1}}$$

where $P_i^{\pm}(z) = \sum_{s\geq 0} P^{\pm}_{i,\pm s}z^{\mp s} = \Theta^{\pm}_{i+1}(z)/\Theta^{\pm}_i(z)$

We denote by $U^0 \subset U_\hbar(L\mathfrak{gl}_n)$ the commutative subalgebra generated by the elements $D_{j,r}$.

7.2 The Yangian $Y_\hbar(\mathfrak{gl}_n)$

The following definition can be found in [25, §3.1]. $Y_\hbar(\mathfrak{gl}_n)$ is the algebra over $\mathbb{C}[\hbar]$ generated by elements $\{e_{i,r}, f_{i,r}, \theta_{j,r}\}_{i\in\mathbf{I}, j\in\mathbf{J}, r\in\mathbb{N}}$ subject to the following relations.[3]

(Y1-gl) For any $j, j' \in \mathbf{J}$ and $r, s \in \mathbb{N}$,

$$[\theta_{j,r}, \theta_{j',s}] = 0$$

(Y2-gl) For any $j \in \mathbf{J}$ and $i \in \mathbf{I}$,

$$\left[\theta_{j,0}, e_{i,s}\right] = c_{ji}e_{i,s}$$
$$\left[\theta_{j,0}, f_{i,s}\right] = -c_{ji}f_{i,s}$$

[3] our conventions are adapted to [15,34] They differ from those of [25] by the permutation $e_{i,r} \leftrightarrow f_{i,r}$ and the relabelling $\theta_{i,r} \leftrightarrow h_{i,r}$.

$$\left[\theta_{j,r+1}, e_{i,s}\right] - \left[\theta_{j,r}, e_{i,s+1}\right] = \hbar c_{ji} e_{i,s} \theta_{j,r}$$
$$\left[\theta_{j,r+1}, f_{i,s}\right] - \left[\theta_{j,r}, f_{i,s+1}\right] = -\hbar c_{ji} \theta_{j,r} f_{i,s}$$

where $c_{ji} = -(\delta_{ji} - \delta_{j,i+1})$.

(Y3-gl) For any $i \in \mathbf{I}$,

$$\left[e_{i,r+1}, e_{i,s}\right] - \left[e_{i,r}, e_{i,s+1}\right] = \hbar(e_{i,r} e_{i,s} + e_{i,s} e_{i,r})$$
$$\left[f_{i,r+1}, f_{i,s}\right] - \left[f_{i,r}, f_{i,s+1}\right] = -\hbar(f_{i,r} f_{i,s} + f_{i,s} f_{i,r})$$

For any $i \in \mathbf{I} \setminus \{n-1\}$ and $r, s \in \mathbb{N}$,

$$\left[e_{i,r+1}, e_{i+1,s}\right] - \left[e_{i,r}, e_{i+1,s+1}\right] = -\hbar e_{i+1,s} e_{i,r}$$
$$\left[f_{i,r+1}, f_{i+1,s}\right] - \left[f_{i,r}, f_{i+1,s+1}\right] = \hbar f_{i,r} f_{i+1,s}$$

(Y4-gl) For any $i, i' \in \mathbf{I}$,

$$[e_{i,r}, f_{i',s}] = \delta_{i,i'} p_{i,r+s}$$

where $p_i(v) = 1 + \hbar \sum_{r \geq 0} p_r v^{-r-1} = \theta_{i+1}(v)\theta_i(v)^{-1}$.

(Y5-gl) For any $i, i' \in \mathbf{I}$ such that $|i - i'| = 1$, and $r_1, r_2, s \in \mathbb{N}$,

$$\left[e_{i,r_1}, \left[e_{i,r_2}, e_{i',s}\right]\right] + \left[e_{i,r_2}, \left[e_{i,r_1}, e_{i',s}\right]\right] = 0$$
$$\left[f_{i,r_1}, \left[f_{i,r_2}, f_{i',s}\right]\right] + \left[f_{i,r_2}, \left[f_{i,r_1}, f_{i',s}\right]\right] = 0$$

For $i, i' \in \mathbf{I}$ such that $|i - i'| > 1$, and $r, s \in \mathbb{N}$

$$\left[e_{i,r}, e_{i',s}\right] = 0 = \left[f_{i,r}, f_{i',s}\right]$$

The Yangian $Y_\hbar(\mathfrak{gl}_n)$ is $\mathbb{N}$-graded by $\deg(e_{i,r}) = \deg(f_{i,r}) = \deg(\theta_{j,r}) = r$ and $\deg(\hbar) = 1$.

7.3 Shift homomorphisms

Let $Y^0 \subset Y_\hbar(\mathfrak{gl}_n)$ be the commutative subalgebra generated by the elements $\{\theta_{j,r}\}$ and $Y^+, Y^- \subset Y_\hbar(\mathfrak{gl}_n)$ the subalgebras generated by Y^0 and the elements $\{e_{i,r}\}$ (resp. $\{f_{i,r}\}$), $i \in \mathbf{I}$, $r \in \mathbb{N}$.

For any $i \in \mathbf{I}$, define, as in Sect. 2.6, a Y^0-linear homomorphism $\sigma_i^\pm$ of $Y^\pm$ by $e_{i',r} \to e_{i',r+\delta_{ii'}}$ (resp. $f_{i',r} \to f_{i',r+\delta_{ii'}}$). The definition of $\sigma_i^\pm$ relies on the PBW theorem for $Y_\hbar(\mathfrak{gl}_n)$, which is proved in [27].

7.4 Alternative generators for Y^0

Define an alternative family of generators $\{d_{j,r}\}_{j\in\mathbf{J},r\in\mathbb{N}}$ of Y^0 by

$$d_j(u) = \hbar \sum_{r\geq 0} d_{j,r} u^{-r-1} = \log(\theta_j(u))$$

Set $B_j(v) = \hbar \sum_{r\geq 0} d_{j,r} \frac{v^r}{r!} \in Y^0[[v]]$. The following commutation relations are proved exactly as their counterparts in Lemma 2.9.

Lemma *The following holds for any $j \in \mathbf{J}$ and $i \in \mathbf{I}$,*

$$\left[B_j(v), e_{i,s}\right] = \frac{1 - e^{-c_{ji}\hbar v}}{v} e^{\sigma_i^+ v}\, e_{i,s}$$

$$\left[B_j(v), f_{i,s}\right] = -\frac{1 - e^{-c_{ji}\hbar v}}{v} e^{\sigma_i^- v}\, f_{i,s}$$

7.5 The operators $\lambda_{\mathbf{i}}^{\pm}(\mathbf{v})$

The following result is analogous to and proved in the same way as, Proposition 2.10.

Proposition *There are operators $\{\lambda_{i;s}^{\pm}\}_{i\in\mathbf{I},s\in\mathbb{N}}$ on Y^0 such that the following holds.*

(1) *For any $\xi \in Y^0$,*

$$e_{i,r}\xi = \sum_{s\geq 0} \lambda_{i;s}^{+}(\xi) e_{i,r+s}$$

$$f_{i,r}\xi = \sum_{s\geq 0} \lambda_{i;s}^{-}(\xi) f_{i,r+s}$$

(2) *The operator $\lambda_i^{\pm}(v) : Y^0 \to Y^0[v]$ given by*

$$\lambda_i^{\pm}(v)(\xi) = \sum_{s\in\mathbb{N}} \lambda_{i;s}^{\pm}(\xi) v^s$$

is an algebra homomorphisms of degree 0 with respect to the $\mathbb{N}$-grading on $Y^0[v]$ extending that on Y^0 by $\deg(v) = 1$.

(3) *The operators $\lambda_i^{\epsilon}(v)$ and $\lambda_{i'}^{\epsilon'}(v')$ commute for any $i, i' \in \mathbf{I}$ and $\epsilon, \epsilon' \in \{\pm\}$. Moreover,*

$$\lambda_i^{+}(v)\lambda_i^{-}(v) = Id$$

(4) *For any* $i \in \mathbf{I}$ *and* $j \in \mathbf{J}$,

$$(\lambda_i^{\pm}(v_1) - 1)B_j(v_2) = \pm \frac{e^{-c_{ji}\hbar v_2} - 1}{v_2} e^{v_1 v_2} \tag{7.1}$$

7.6

Let $\{g_i^{\pm}(u)\}_{i \in \mathbf{I}}$ be a collection of elements in $\widehat{Y^0[u]}$. Define, as in Sect. 3.1, an assignment $\Phi : \{E_{i,r}, F_{i,r}, D_{j,r}\} \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$ by

$$\begin{aligned}
\Phi(D_{j,0}) &= \theta_{j,0} \\
\Phi(D_{j,r}) &= \frac{B_j(r)}{q - q^{-1}} \text{ for } r \neq 0 \\
\Phi(E_{i,k}) &= \mathrm{e}^{k\sigma_i^+} g_i^+(\sigma_i^+)\, e_{i,0} \\
\Phi(F_{i,k}) &= \mathrm{e}^{k\sigma_i^-} g_i^-(\sigma_i^-)\, f_{i,0}
\end{aligned}$$

and denote the restriction of Φ to U^0 by Φ^0. For any $i \in \mathbf{I}$, set

$$\begin{aligned}
\xi_i(u) &= 1 + \hbar \sum_{r \geq 0} \xi_{i,r} u^{-r-1} = \theta_{i+1}(u)\theta_i(u)^{-1} \in Y^0[[u^{-1}]] \\
P_i^{\pm}(z) &= \sum_{s \geq 0} P_{i,\pm s}^{\pm} z^{\mp s} = \Theta_{i+1}^{\pm}(z)\Theta_i^{\pm}(z)^{-1} \in U^0[[z^{\mp 1}]]
\end{aligned}$$

Theorem *The assignment* Φ *extends to an algebra homomorphism if and only if the following conditions hold.*

(A) *For any* $i, i' \in \mathbf{I}$,

$$g_i^+(u)\lambda_i^+(u)(g_{i'}^-(v)) = g_{i'}^-(v)\lambda_{i'}^-(v)(g_i^+(u))$$

(B) *For any* $i \in \mathbf{I}$ *and* $k \in \mathbb{Z}$,

$$e^{ku} g_i^+(v)\lambda_i^+(v)(g_i^-(v))|_{v^m = \xi_{i,m}} = \Phi^0\left(\frac{P_{i,k}^+ - P_{i,k}^-}{q - q^{-1}}\right)$$

(C0) *For any* $i, i' \in \mathbf{I}$ *such that* $|i - i'| > 1$,

$$g_i^{\pm}(u)\lambda_i^{\pm}(u)(g_{i'}^{\pm}(v)) = g_{i'}^{\pm}(v)\lambda_{i'}^{\pm}(v)(g_i^{\pm}(u))$$

(C1) *For any* $i \in \mathbf{I}$

$$g_i^{\pm}(u)\lambda_i^{\pm}(u)(g_i^{\pm}(v))\frac{e^u - e^{v \pm \hbar}}{u - v \mp \hbar} = g_i^{\pm}(v)\lambda_i^{\pm}(v)(g_i^{\pm}(u))\frac{e^v - e^{u \pm \hbar}}{v - u \mp \hbar}$$

(C2) *For any* $i \in \mathbf{I}\setminus\{n-1\}$,

$$g_i^{\pm}(u)\lambda_i^{\pm}(u)(g_{i+1}^{\pm}(v))\left(\frac{e^u-e^v}{u-v}\right)^{\pm 1}=g_{i+1}^{\pm}(v)\lambda_{i+1}^{\pm}(v)(g_i^{\pm}(u))\left(\frac{e^{u-\hbar/2}-e^{v+\hbar/2}}{u-v-\hbar}\right)^{\pm 1}$$

The proof of Theorem 7.6 is given in Sects. 7.7–7.11. It follows the same lines as that of Theorem 3.4, with the exception of the q-Serre relations (QL5-gl) which are proved directly.

7.7

A proof similar to that of Lemmas 3.5 and 3.6 yields the following

(1) Φ is compatible with the relation (QL4-gl) if, and only if (A) and (B) hold.
(2) Φ is compatible with the relation (QL3-gl) if, and only if (C0)–(C2) hold.

By virtue of condition (C0), Φ is compatible with the q-Serre relations (QL5-gl) whenever $|i-i'|>1$. We therefore need to only consider the case $|i-i'|=1$. We shall in fact restrict to $i'=i+1$ since the case $i'=i-1$ is dealt with similarly.

7.8

The essential ingredient is the following analogue of Lemma 8.4. We leave it to the reader to carry out the construction of the auxiliary algebra $\overline{Y}$ (see Sect. 8.2), the operators $\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)}$ and $\overline{\sigma}_{i'}$ on $\overline{Y}_{2\alpha_i+\alpha_{i'}}$ (Sect. 8.3) and the map $p_{ii'}:\overline{Y}_{2\alpha_i+\alpha_{i'}}\to Y_\hbar(\mathfrak{gl}_n)$.

Lemma *The kernel of $p_{ii'}$ is the $\mathbb{C}[\hbar]$-linear span of the following elements*

(1) *For any* $A(u_1,u_2,v)\in\overline{Y}^0[u_1,u_2,v]$

$$A(\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)},\overline{\sigma}_{i'})\left((\overline{\sigma}_{i,(2)}-\overline{\sigma}_{i'})\overline{e}_{i,0}^2\overline{e}_{i',0}-(\overline{\sigma}_{i,(2)}-\overline{\sigma}_{i'}-\hbar)\overline{e}_{i,0}\overline{e}_{i',0}\overline{e}_{i,0}\right)$$
$$A(\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)},\overline{\sigma}_{i'})\left((\overline{\sigma}_{i,(1)}-\overline{\sigma}_{i'})\overline{e}_{i,0}\overline{e}_{i',0}\overline{e}_{i,0}-(\overline{\sigma}_{i,(1)}-\overline{\sigma}_{i'}-\hbar)\overline{e}_{i',0}\overline{e}_{i,0}^2\right)$$

(2) *For any* $B(u_1,u_2,v)=B(u_2,u_1,v)\in\overline{Y}^0[u_1,u_2,v]$

$$B(\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)},\overline{\sigma}_{i'})(\overline{\sigma}_{i,(1)}-\overline{\sigma}_{i,(2)}-\hbar)\overline{e}_{i,0}^2\overline{e}_{i',0}$$
$$B(\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)},\overline{\sigma}_{i'})(\overline{\sigma}_{i,(1)}-\overline{\sigma}_{i,(2)}-\hbar)\overline{e}_{i',0}\overline{e}_{i,0}^2$$

(3) *For any* $B(u_1,u_2,v)=B(u_2,u_1,v)\in\overline{Y}^0[u_1,u_2,v]$

$$B(\overline{\sigma}_{i,(1)},\overline{\sigma}_{i,(2)},\overline{\sigma}_{i'})\left(\overline{e}_{i,0}^2\overline{e}_{i',0}-2\overline{e}_{i,0}\overline{e}_{i',0}\overline{e}_{i,0}+\overline{e}_{i',0}\overline{e}_{i,0}^2\right)$$

Corollary *The kernel of* $p_{ii'}$ *is stable under the action of* $A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'})$*, for any* $A(u_1, u_2, v) = A(u_2, u_1, v) \in \overline{Y}^0[u_1, u_2, v]$.

Remark In the next sections, we use the following convention for notational convenience: for each $\overline{X} \in \overline{Y}_{2\alpha_i+\alpha_{i'}}$ and $X = p_{ii'}(\overline{X})$, we set

$$A(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})(X) = p_{i,i'}\left(A(\overline{\sigma}_{i,(1)}, \overline{\sigma}_{i,(2)}, \overline{\sigma}_{i'})(\overline{X})\right)$$

7.9

We shall only prove the q-Serre relations for the case of the E's and consequently drop the superscript $+$. We need to show that the following holds for any $k_1, k_2, l \in \mathbb{Z}$

$$\begin{aligned}\Phi(E_{i,k_1})\Phi(E_{i,k_2})\Phi(E_{j,l}) - (q+q^{-1})\Phi(E_{i,k_1})\Phi(E_{j,l})\Phi(E_{i,k_2})\\ +\Phi(E_{j,l})\Phi(E_{i,k_1})\Phi(E_{i,k_2}) + (k_1 \leftrightarrow k_2) = 0\end{aligned}$$

As in Sect. 8.5, an application of Corollary 7.8 shows that this reduces to showing that

$$\Phi(E_{i,0})^2\Phi(E_{j,0}) - (q+q^{-1})\Phi(E_{i,0})\Phi(E_{j,0})\Phi(E_{i,0}) + \Phi(E_{j,0})\Phi(E_{i,0})^2 = 0$$

7.10

With Corollary 7.8 in mind, we seek to factor a common symmetric function out of each of the above summands. This is achieved by the following result.

Lemma *There exists* $H(u_1, u_2, v) \in \overline{Y}^0[[u_1, u_2, v]]$ *symmetric in* $u_1 \leftrightarrow u_2$*, such that*

$$\begin{aligned}\Phi(E_{i,0})^2\Phi(E_{i',0}) &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_0(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i,0}e_{i',0}\\ \Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_1(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i,0}e_{i',0}e_{i,0}\\ \Phi(E_{i',0})\Phi(E_{i,0})^2 &= H(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})\mathcal{P}_2(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i'})e_{i',0}e_{i,0}e_{i,0}\end{aligned}$$

where $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2 \in \mathbb{C}[[u_1, u_2, v]]$ *are given in terms of the function*

$$P(x, y) = \frac{e^x - e^y}{x - y} \in \mathbb{C}[[x, y]]^{\mathfrak{S}_2}$$

by

$$\begin{aligned}\mathcal{P}_0 &= P(u_1 + \hbar, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2 - \hbar/2, v + \hbar/2)\\ \mathcal{P}_1 &= P(u_1 + \hbar, u_2)P(u_1 - \hbar/2, v + \hbar/2)P(u_2, v)\\ \mathcal{P}_2 &= P(u_1 + \hbar, u_2)P(u_1, v)P(u_2, v)\end{aligned}$$

Proof Define $G_{ab}(x,y) \in \overline{Y}^0[[x,y]]$ by $\lambda_a(x)(g_b(y)) = g_b(y)G_{ab}(x,y)$. Then, in obvious notation,

$$\begin{aligned}
&\Phi(E_{a,0})\Phi(E_{b,0})\Phi(E_{c,0})\\
&\quad= g_a(\sigma_a)e_{a,0}g_b(\sigma_b)e_{b,0}g_c(\sigma_c)e_{c,0}\\
&\quad= g_a(\sigma_a)\lambda_a(\sigma_a)(g_b(\sigma_b))\lambda_a(\sigma_a)\circ\lambda_b(\sigma_b)(g_c(\sigma_c))e_{a,0}e_{b,0}e_{c,0}\\
&\quad= g_a(\sigma_a)g_b(\sigma_b)g_c(\sigma_c)G_{ab}(\sigma_a,\sigma_b)G_{ac}(\sigma_a,\sigma_c)\lambda_a(\sigma_a)(G_{bc}(\sigma_b,\sigma_c))e_{a,0}e_{b,0}e_{c,0}
\end{aligned}$$

We record for later use the symmetry in the interchange $a \leftrightarrow b$ of the term

$$\begin{aligned}
G_{ac}(\sigma_a,\sigma_c)\lambda_a(\sigma_a)(G_{bc}(\sigma_b,\sigma_c)) &= \lambda_a(\sigma_a)\circ\lambda_b(\sigma_b)(g_c(\sigma_c))/g_c(\sigma_c)\\
&= G_{bc}(\sigma_b,\sigma_c)\lambda_b(\sigma_b)(G_{ac}(\sigma_a,\sigma_c)) \qquad (7.2)
\end{aligned}$$

where the second equality follows from the commutativity of $\lambda_a(\sigma_a)$ and $\lambda_b(\sigma_b)$.

Set now $F = g_i(\sigma_{i,1})g_i(\sigma_{i,2})g_{i'}(\sigma_{i'}) \in \overline{Y}^0[[\sigma_{i,1},\sigma_{i,2},\sigma_{i'}]]^{\mathfrak{S}_2}$. Then, the above yields

$$\begin{aligned}
\Phi(E_{i,0})^2\Phi(E_{i',0}) &= F\, G_{ii}(\sigma_{i,1},\sigma_{i,2})G_{ii'}(\sigma_{i,1},\sigma_{i'})\lambda_i(\sigma_{i,1})(G_{ii'}(\sigma_{i,2},\sigma_{i'}))e_{i,0}^2e_{i',0}\\
\Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) &= F\, G_{ii'}(\sigma_{i,1},\sigma_{i'})G_{ii}(\sigma_{i,1},\sigma_{i,2})\lambda_i(\sigma_{i,1})(G_{i'i}(\sigma_{i'},\sigma_{i,2}))\\
&\qquad e_{i,0}e_{i',0}e_{i,0}\\
\Phi(E_{i',0})\Phi(E_{i,0})^2 &= F\, G_{i'i}(\sigma_{i'},\sigma_{i,1})G_{i'i}(\sigma_{i'},\sigma_{i,2})\lambda_{i'}(\sigma_{i'})(G_{ii}(\sigma_{i,1},\sigma_{i,2}))e_{i',0}e_{i,0}^2
\end{aligned}$$

We claim that $G_{ii}(u_1,u_2) = \overline{G}_{ii}(u_1,u_2)P(u_1+\hbar,u_2)$, where $\overline{G}$ is symmetric in u_1,u_2. Indeed, by condition (C1)

$$G_{ii}(u_1,u_2)P(u_1,u_2+\hbar) = G_{ii}(u_2,u_1)P(u_2,u_1+\hbar)$$

whence the result with $\overline{G}_{ii}(u_1,u_2) = G_{ii}(u_1,u_2)/P(u_1+\hbar,u_2)$. It follows that

$$\Phi(E_{i,0})^2\Phi(E_{i',0}) = \overline{H}(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})P(\sigma_{i,1}+\hbar,\sigma_{i,2})e_{i,0}^2e_{i',0}$$

where

$$\begin{aligned}
\overline{H}(u_1,u_2,v) &= g_i(u_1)g_i(u_2)g_{i'}(v)\overline{G}_{ii}(u_1,u_2)G_{ii'}(u_1,v)\lambda_i(u_1)(G_{ii'}(u_2,v))\\
&\in Y^0[[u_1,u_2,v]]
\end{aligned}$$

is symmetric in u_1,u_2 by (7.2).

Next, assuming that $i' = i+1$, condition (C2) yields

$$G_{ii'}(u,v)P(u,v) = G_{i'i}(v,u)P(u-\hbar/2,v+\hbar/2)$$

so that

$$\Phi(E_{i,0})\Phi(E_{i',0})\Phi(E_{i,0}) \\ = \overline{H}(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})P(\sigma_{i,1}+\hbar,\sigma_{i,2})\frac{P(\sigma_{i,2},\sigma_{i'})}{P(\sigma_{i,2}-\hbar/2,\sigma_{i'}+\hbar/2)}e_{i,0}e_{i',0}e_{i,0}$$

Finally, using (7.2) again, with $a=i, b=i', c=i$ and the previous calculation yields

$$\Phi(E_{i',0})\Phi(E_{i,0})^2 = \overline{H}(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})P(\sigma_{i,1}+\hbar,\sigma_{i,2}) \\ \frac{P(\sigma_{i,1},\sigma_{i'})}{P(\sigma_{i,1}-\hbar/2,\sigma_{i'}+\hbar/2)}\frac{P(\sigma_{i,2},\sigma_{i'})}{P(\sigma_{i,2}-\hbar/2,\sigma_{i'}+\hbar/2)}e_{i',0}e_{i,0}^2$$

as claimed. □

7.11

By Lemma 7.10 and Corollary 7.8, we are reduced to proving the following

$$\mathcal{S}^q = \mathcal{P}_0(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i,0}^2e_{i',0} - (q+q^{-1})\mathcal{P}_1(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i,0}e_{i',0}e_{i,0} \\ + \mathcal{P}_2(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i',0}e_{i,0}^2 = 0$$

Step 1. We first observe that

$$P(u_1+\hbar,u_2) - \frac{1+e^{\hbar}}{2}P(u_1,u_2) \in (u_1-u_2-\hbar)\mathbb{C}[[\hbar,u_1,u_2]]^{\mathfrak{S}_2}$$

This allows us to use (2) of Lemma 7.8 to obtain

$$\mathcal{S}^q = \mathcal{P}_0'(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i,0}^2e_{i',0} - 2\mathcal{P}_1'(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i,0}e_{i',0}e_{i,0} \\ + \mathcal{P}_2'(\sigma_{i,1},\sigma_{i,2},\sigma_{i'})e_{i',0}e_{i,0}^2$$

where

$$\mathcal{P}_0' = e^{\hbar/2}P(u_1,u_2)P(u_1-\hbar/2,v+\hbar/2)P(u_2-\hbar/2,v+\hbar/2)$$
$$\mathcal{P}_1' = P(u_1+\hbar,u_2)P(u_1-\hbar/2,v+\hbar/2)P(u_2,v) = \mathcal{P}_1$$
$$\mathcal{P}_2' = e^{\hbar/2}P(u_1,u_2)P(u_1,v)P(u_2,v)$$

Step 2. We use next (3) of Lemma 7.8 with $B=\mathcal{P}_2'$ to get

$$\mathcal{S}^q = (\mathcal{P}_0'-\mathcal{P}_2')e_{i,0}^2e_{i',0} - 2(\mathcal{P}_1'-\mathcal{P}_2')e_{i,0}e_{i',0}e_{i,0}$$

One can easily check that $\mathcal{P}_1' - \mathcal{P}_2'$ is divisible by $u_2 - v - \hbar$, which together with (1) of Lemma 7.8, with $A = 2\dfrac{\mathcal{P}_1' - \mathcal{P}_2'}{u_2 - v - \hbar}$, yields

$$\mathcal{S}^q = \left(\mathcal{P}_0' - \mathcal{P}_2' - 2\frac{\mathcal{P}_1' - \mathcal{P}_2'}{u_2 - v - \hbar}(u_2 - v)\right) e_{i,0}^2 e_{i',0}$$

Step 3. Finally we can directly verify that the function

$$\mathcal{F} = \mathcal{P}_0' - \mathcal{P}_2' - 2\frac{\mathcal{P}_1' - \mathcal{P}_2'}{u_2 - v - \hbar}(u_2 - v)$$

is divisible by $u_1 - u_2 - \hbar$. Moreover, the quotient $\dfrac{\mathcal{F}}{u_1 - u_2 - \hbar}$ is symmetric in u_1, u_2. This allows us to use (2) of Lemma 7.8 to conclude that $\mathcal{S}^q = 0$.

7.12 The variety $\mathcal{F}$

Fix integers $1 \le n \le d \in \mathbb{N}$, let

$$\mathcal{F} = \left\{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^d\right\}$$

be the variety of n-step flags in $\mathbb{C}^d$, and $T^*\mathcal{F}$ its cotangent bundle. We describe below the $GL_d(\mathbb{C}) \times \mathbb{C}^\times$-equivariant K-theory and cohomology of $T^*\mathcal{F}$ following [15,34].

The connected components of $\mathcal{F}$ are parametrised by the set $\mathcal{P}$ of partitions of $[1, d]$ into n intervals, i.e.,

$$\mathcal{P} = \{\mathbf{d} = (0 = d_0 \le d_1 \le \cdots \le d_n = d)\}$$

where $\mathbf{d} \in \mathcal{P}$ labels the component $\mathcal{F}_\mathbf{d}$ consisting of flags such that $\dim V_k = d_k$. The symmetric group $\mathfrak{S}_d$ acts on the rings

$$S = \mathbb{C}[q^{\pm 1}, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$
$$R = \mathbb{C}[\hbar, x_1, \ldots, x_d]$$

by permuting the variables $X_1, \ldots, X_d$ and $x_1, \ldots, x_d$ and fixing $q, \hbar$ respectively. For any $\mathbf{d} \in \mathcal{P}$, denote by

$$\mathfrak{S}(\mathbf{d}) = \mathfrak{S}_{d_1 - d_0} \times \cdots \times \mathfrak{S}_{d_n - d_{n-1}} \subset \mathfrak{S}_d$$

the subgroup preserving the corresponding partition. Then, the following holds

$$K^{GL_d(\mathbb{C})\times\mathbb{C}^\times}(T^*\mathcal{F}) \cong \bigoplus_{\mathbf{d}\in\mathcal{P}} S^{\mathfrak{S}(\mathbf{d})}$$
$$H_{GL_d(\mathbb{C})\times\mathbb{C}^\times}(T^*\mathcal{F}) \cong \bigoplus_{\mathbf{d}\in\mathcal{P}} R^{\mathfrak{S}(\mathbf{d})}$$

where $K^{\mathbb{C}^\times}(pt) = \mathbb{C}[q, q^{-1}]$ and $H_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$.

7.13

For any partition $\mathbf{d} \in \mathcal{P}$ and $i \in \mathbf{I}$, set

$$\mathbf{d}_i^{\pm} = (0 = d_0 \le \cdots \le d_{i-1} \le d_i \pm 1 \le d_{i+1} \le \cdots \le d_n = d)$$

if the right-hand side makes sense as a partition.

If $\mathbf{d}, \mathbf{d}' \in \mathcal{P}$ are two partitions, and P is one of the rings R, S, we denote by $\sigma(\mathbf{d}, \mathbf{d}')$ the symmetrisation operator

$$\sigma(\mathbf{d}, \mathbf{d}') : P^{\mathfrak{S}(\mathbf{d})\cap\mathfrak{S}(\mathbf{d}')} \to P^{\mathfrak{S}(\mathbf{d}')}, \qquad \sigma(\mathbf{d}, \mathbf{d}')(p) = \sum_{\tau\in\mathfrak{S}(\mathbf{d}')/\mathfrak{S}(\mathbf{d})\cap\mathfrak{S}(\mathbf{d}')} \tau(p)$$

7.14 $U_\hbar(L\mathfrak{gl}_n)$-action [15,34]

Consider the following operators acting on

$$S(\mathcal{P}) = \bigoplus_{\mathbf{d}\in\mathcal{P}} S^{\mathfrak{S}(\mathbf{d})}$$

(1) For any $j \in \mathbf{J}$, $\Psi_U(\Theta_j^{\pm}(z))$ acts on $S^{\mathfrak{S}(\mathbf{d})}$ as multiplication by

$$\prod_{k=1}^{d_{j-1}} \frac{qz - q^{-1}X_k}{z - X_k} \prod_{k=d_j+1}^{d} \frac{z - X_k}{q^{-1}z - qX_k} \in S^{\mathfrak{S}(\mathbf{d})}[[z^{\mp 1}]]$$

(2) For any $i \in \mathbf{I}$, the operators

$$\Psi_U(E_i(z)) : S^{\mathfrak{S}(\mathbf{d})} \to S^{\mathfrak{S}(\mathbf{d}_i^+)}[[z, z^{-1}]]$$
$$\Psi_U(F_i(z)) : S^{\mathfrak{S}(\mathbf{d})} \to S^{\mathfrak{S}(\mathbf{d}_i^-)}[[z, z^{-1}]]$$

act by 0 if $\mathbf{d}_i^{\pm}$ is not defined, and by

$$\Psi_U(E_i(z))p = \sigma(\mathbf{d}, \mathbf{d}_i^+) \left(\delta(X_{d_i+1}/z) \prod_{k \in I_i} \frac{qX_{d_i+1} - q^{-1}X_k}{X_{d_i+1} - X_k} p \right)$$

$$\Psi_U(F_i(z))p = \sigma(\mathbf{d}, \mathbf{d}_i^-) \left(\delta(X_{d_i}/z) \prod_{k \in I_{i+1}} \frac{q^{-1}X_{d_i} - qX_k}{X_{d_i} - X_k} p \right)$$

otherwise, where I_i is the interval $[d_{i-1} + 1, \ldots, d_i]$.

The following result is due to Ginzburg and Vasserot and is proved in [34, §2.2].

Theorem *The assignment Ψ_U extends to an algebra homomorphism*

$$\Psi_U : U_\hbar(L\mathfrak{gl}_n) \to End_{\mathbb{C}[q,q^{-1}]}(S(\mathcal{P}))$$

7.15 $Y_\hbar(\mathfrak{gl}_n)$-action

Consider the following operators acting on

$$R(\mathcal{P}) = \bigoplus_{\mathbf{d} \in \mathcal{P}} R^{\mathfrak{S}(\mathbf{d})}$$

(1) For any $j \in \mathbf{J}$, $\Psi_Y(\theta_j(u))$ acts on $R^{\mathfrak{S}(\mathbf{d})}$ as multiplication by

$$\prod_{k=1}^{d_{j-1}} \frac{u - x_k + \hbar}{u - x_k} \prod_{k=d_j+1}^{d} \frac{u - x_k}{u - x_k - \hbar} \in R^{\mathfrak{S}(\mathbf{d})}[[u^{-1}]]$$

(2) For any $i \in \mathbf{I}$,

$$\Psi_Y(e_i(u)) : R^{\mathfrak{S}(\mathbf{d})} \to R^{\mathfrak{S}(\mathbf{d}_i^+)}[[u^{-1}]]$$

$$\Psi_Y(f_i(u)) : R^{\mathfrak{S}(\mathbf{d})} \to R^{\mathfrak{S}(\mathbf{d}_i^-)}[[u^{-1}]]$$

act as zero if $\mathbf{d}_i^{\pm}$ is not defined, and by

$$\Psi_Y(e_i(u))p = \hbar\sigma(\mathbf{d}, \mathbf{d}_i^+) \left(\frac{1}{u - x_{d_i+1}} \prod_{k \in I_i} \frac{x_{d_i+1} - x_k + \hbar}{x_{d_i+1} - x_k} p \right)$$

$$\Psi_Y(f_i(u))p = \hbar\sigma(\mathbf{d}, \mathbf{d}_i^-) \left(\frac{1}{u - x_{d_i}} \prod_{k \in I_{i+1}} \frac{x_{d_i} - x_k - \hbar}{x_{d_i} - x_k} p \right)$$

otherwise.

The following result is proved in a similar way to Theorem 7.14

Proposition *The assignment* Ψ_Y *extends to an algebra homomorphism*

$$\Psi_Y : Y_\hbar(\mathfrak{gl}_n) \to End_{\mathbb{C}[\hbar]}(R(\mathcal{P}))$$

Remark The above formulae are degenerations of those of the previous section obtained by setting $z = \mathrm{e}^{tu}$, $q = \mathrm{e}^{t\hbar/2}$, $X_k = \mathrm{e}^{tx_k}$ and letting $t \to 0$.

7.16

Lemma *The homomorphism* Ψ_Y *maps the centre* $\mathcal{Z}$ *of* $Y_\hbar(\mathfrak{gl}_n)$ *surjectively to* $\mathbb{C}[\hbar, x_1, \ldots, x_d]^{\mathfrak{S}_d}$*. In particular, there exists an element*

$$\Delta(u) = 1 + \hbar \sum_{r\geq 0} \Delta_r u^{-r-1} \in \mathcal{Z}[[u^{-1}]]$$

such that

$$\Psi_Y(\Delta(u)) = \prod_{k=1}^{d} \frac{u - x_k - \hbar}{u - x_k}$$

Proof By [25, Cor. 1.11.8], $\mathcal{Z}$ is generated by the coefficients of the element

$$qdet(u) = \theta_1(u)\theta_2(u-\hbar)\cdots\theta_n(u-(n-1)\hbar) \in Y_\hbar(\mathfrak{gl}_n)[[u^{-1}]]$$

It readily follows from 7.15 that

$$\Psi_Y(qdet(u)) = \prod_{k=1}^{d} \frac{u - x_k}{u - x_k - (n-1)\hbar}$$

By (2.7), $L(v) = B(\log(qdet(u))) \in \mathcal{Z}[[v]]$ therefore satisfies

$$\Psi_Y(L(v)) = \sum_{k=1}^{d} \frac{\mathrm{e}^{(x_k+(n-1)\hbar)v} - \mathrm{e}^{x_k v}}{v} = \sum_{r\geq 1}\Big(p_r(\{x_k + (n-1)\hbar\}) - p_r(\{x_k\})\Big)\frac{v^{r-1}}{r!}$$

which yields the surjectivity since the power sums $p_r(x_1, \ldots, x_d) = \sum_k x_k^r$ generate $\mathbb{C}[x_1, \ldots, x_d]^{\mathfrak{S}_d}$. □

7.17

We will need the following

Lemma *For any* $i \in \mathbf{I}$*, there exists* $\mathrm{Td}_i^{\pm}(v) = \sum_{r\geq 0} \mathrm{Td}_{i,r}^{\pm} v^r \in \widehat{Y^0[v]}$ *such that*

$$\Psi_Y\left(\mathrm{Td}_i^{+}(v)\right) = \prod_{k\in I_i} \frac{v - x_k}{1 - e^{-v+x_k}} \frac{1 - e^{-v+x_k-\hbar}}{v - x_k + \hbar}$$

$$\Psi_Y\left(\mathrm{Td}_i^-(v)\right) = \prod_{k\in I_{i+1}} \frac{v-x_k}{1-e^{-v+x_k}} \frac{1-e^{-v+x_k+\hbar}}{v-x_k-\hbar}$$

The proof of this lemma is given in Sect. 7.20.

7.18 A compatible assignment

Let $\Phi : \{E_{i,0}, F_{i,0}, D_{j,r}\}_{i\in\mathbf{I}, j\in\mathbf{J}, r\in\mathbb{Z}} \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$ be the assignment defined by

$$\begin{aligned}
\Phi(D_{j,0}) &= \theta_{j,0}\\
\Phi(D_{j,r}) &= \left.\frac{B_j(v)}{q-q^{-1}}\right|_{v=r}\\
\Phi(E_{i,0}) &= \sum_{s\geq 0} e_{i,s}\mathrm{Td}_{i,s}^+ \, q^{-\Delta_0-\theta_{i,0}}\\
\Phi(F_{i,0}) &= \sum_{s\geq 0} f_{i,s}\mathrm{Td}_{i,s}^- \, q^{\Delta_0+\theta_{i+1,0}}
\end{aligned}$$

where Δ_0 is given by Lemma 7.16. Extend Φ to the generators $E_{i,r}$, $F_{i,r}$, $r \in \mathbb{Z}$ by defining, as in Sect. 3.3,

$$\mathrm{Td}_i^{\pm,(r)}(v) = \sum_{m\geq 0} \mathrm{Td}_{i,m}^{\pm,(r)} v^m = \mathrm{e}^{rv}\mathrm{Td}_i^\pm(v) \in \widehat{Y^0}[[v]]$$

and setting

$$\begin{aligned}
\Phi(E_{i,r}) &= \sum_{s\geq 0} e_{i,s}\mathrm{Td}_{i,s}^{+,(r)} \, q^{-\Delta_0-\theta_{i,0}}\\
\Phi(F_{i,r}) &= \sum_{s\geq 0} f_{i,s}\mathrm{Td}_{i,s}^{-,(r)} \, q^{\Delta_0+\theta_{i+1,0}}
\end{aligned}$$

7.19

Let $\widehat{R(\mathcal{P})}$ be the completion with respect to the $\mathbb{N}$-grading given by $\deg(x_k) = \deg(\hbar) = 1$. Define a homomorphism ch : $S(\mathcal{P}) \to \widehat{R(\mathcal{P})}$ mapping each $S^{\mathfrak{S}(\mathbf{d})}$ to $\widehat{R^{\mathfrak{S}(\mathbf{d})}}$ by

$$q \longmapsto \mathrm{e}^{\hbar/2} \qquad \text{and} \qquad X_k \longmapsto \mathrm{e}^{x_k}$$

Theorem *The assignment Φ above intertwines the geometric realisations of $U_\hbar(L\mathfrak{gl}_n)$ and $Y_\hbar(\mathfrak{gl}_n)$ on $S(\mathcal{P})$ and $R(\mathcal{P})$ respectively. In other words, the following holds for any $X \in \{E_{i,r}, F_{i,r}, D_{j,r}\}_{i\in\mathbf{I}, j\in\mathbf{J}, r\in\mathbb{N}}$ and $\pi \in S(\mathcal{P})$.*

$$ch(X \cdot \pi) = \Phi(X) \cdot ch(\pi)$$

Proof Consider first the case $X = D_{j,r}$, $j \in \mathbf{J}$, $r \in \mathbb{Z}$. By definition of Ψ_U and Ψ_Y, $D_{j,0} = 2/\hbar \log(\Theta_{j,0})$ and $\theta_{j,0}$ act on $S^{\mathfrak{S}(\mathbf{d})}$ and $R^{\mathfrak{S}(\mathbf{d})}$ respectively as multiplication by $d - (d_j - d_{j-1})$. Further, (2.7) yields

$$\Psi_Y(B_j(v)) = \frac{1}{v}\left((1-e^{-\hbar v})\sum_{k=1}^{d_{j-1}} e^{x_k v} + (e^{\hbar v}-1)\sum_{k=d_j+1}^{d} e^{x_k v}\right)$$

Similarly, taking log in

$$\Psi_U\left(\exp\left((q-q^{-1})\sum_{s\geq 1} D_{j,s} z^{-s}\right)\right) = \prod_{k=1}^{d_{j-1}} \frac{z-q^{-2}X_k}{z-X_k} \prod_{k=d_j+1}^{d} \frac{z-X_k}{z-q^2X_k}$$

yields

$$\Psi_U((q-q^{-1})D_{j,r}) = \frac{1}{r}\left((1-q^{-2r})\sum_{k=1}^{d_{j-1}} X_j^r + (q^{2r}-1)\sum_{k=d_j+1}^{d} X_k^r\right)$$

Thus, $\mathrm{ch}(\Psi_U(D_{j,r})\pi) = \Psi_Y\left(B_j(r)\pi/(q-q^{-1})\right)$ for any $\pi \in S^{\mathfrak{S}(\mathbf{d})}$.

We turn next to $X = E_{i,0}$. Let $\pi \in S^{\mathfrak{S}(\mathbf{d})}$ and set $p = \mathrm{ch}(\pi) \in R^{\mathfrak{S}(\mathbf{d})}$. Since Δ_0 acts on $R^{\mathfrak{S}(\mathbf{d})}$ as multiplication by $-d$ by Lemma 7.16, and $\theta_{j,0}$ acts as multiplication by $d - (d_j - d_{j-1})$, we get

$$\begin{aligned}
\Phi(E_{i,0})(p) &= \sum_{s\geq 0} e_{i,s}\left(\mathrm{Td}_{i,s}^{+}\, p\right) q^{d_j - d_{j-1}} \\
&= \sigma(\mathbf{d}, \mathbf{d}_i^{+})\left(\sum_{s\geq 0} \mathrm{Td}_{i,s}^{+} x_{d_i+1}^{s} p \prod_{k\in I_i} \frac{x_{d_i+1}-x_k+\hbar}{x_{d_i+1}-x_k}\right) q^{d_j-d_{j-1}} \\
&= \sigma(\mathbf{d}, \mathbf{d}_i^{+})\left(\mathrm{Td}_i^{+}(x_{d_i+1})\, p \prod_{k\in I_i} \frac{x_{d_i+1}-x_k+\hbar}{x_{d_i+1}-x_k}\right) q^{d_j-d_{j-1}} \\
&= \sigma(\mathbf{d}, \mathbf{d}_i^{+})\left(p \prod_{k\in I_i} \frac{e^{x_{d_i+1}}-e^{x_k-\hbar}}{e^{x_{d_i+1}}-e^{x_k}}\right) q^{d_j-d_{j-1}} \\
&= \sigma(\mathbf{d}, \mathbf{d}_i^{+})\left(p\mathrm{ch} \prod_{k\in I_i} \frac{qX_{d_i+1}-q^{-1}X_k}{X_{d_i+1}-X_k}\right) \\
&= \mathrm{ch}\left(E_{i,0}\,\pi\right)
\end{aligned}$$

The proof for the rest of the generators is identical. □

7.20 Proof of Lemma 7.17

Let $\Delta(u)$ be the formal series given in Lemma 7.16, and set

$$\mathfrak{z}(u) = \hbar \sum_{r \geq 0} \mathfrak{z}_r u^{-r-1} = \log(\Delta(u))$$

For any $j \in \mathbf{J}$, define $y_j(u) \in Y^0[[u^{-1}]]$ by

$$y_j(u) = \mathfrak{z}(u + (j-1)\hbar) + d_j(u) + \sum_{s=1}^{j-1} \big(d_{j-s}(u + s\hbar) - d_{j-s}(u + (s-1)\hbar)\big) \tag{7.3}$$

A computation similar to the one given in Sect. 7.19 shows that, for any $j \in \mathbf{J}$,

$$\Psi_Y\left(B(y_j(u))\right) = \frac{1 - e^{\hbar v}}{v} \sum_{k \in I_j} e^{x_k v} \tag{7.4}$$

Set now[4]

$$J(v) = \log\left(\frac{v}{1 - e^{-v}}\right) \in \mathbb{Q}[[v]]$$

and, for any $i \in \mathbf{I}$, define

$$\mathrm{td}_i^+(v) = B(y_i(-\partial))J'(v + \hbar) \tag{7.5}$$

$$\mathrm{td}_i^-(v) = -B(y_{i+1}(-\partial))J'(v) \tag{7.6}$$

where $\partial = d/dv$. We claim that $\mathrm{Td}_i^{\pm}(u) = \exp(td_i^{\pm}(u))$ satisfy the conditions of the Lemma. By (7.4) we have

$$\Psi_Y\left(td_i^+(v)\right) = \left(\frac{1 - e^{-\hbar\partial}}{-\partial} \sum_{k \in I_i} e^{-x_k \partial}\right) \partial J(v + \hbar)$$

---

[4] Note the difference between $J(v)$ and the function $G(v)$ used in Sect. 4.1 for constructing the solutions for simple Lie algebras: $J(v) = G(v) + \frac{v}{2}$.

Using $e^{-p\partial} f(v) = f(v-p)$, we get

$$\begin{aligned}\Psi_Y(td_i^+(v)) &= \sum_{k\in I_i} J(v-x_k) - J(v-x_k+\hbar) \\ &= \sum_{k\in I_i} \log\left(\frac{v-x_k}{1-e^{-v+x_k}}\frac{1-e^{-v+x_k-\hbar}}{v-x_k+\hbar}\right) \\ &= \log\left(\prod_{k\in I_i}\frac{v-x_k}{1-e^{-v+x_k}}\frac{1-e^{-v+x_k+\hbar}}{v-x_k+\hbar}\right)\end{aligned}$$

The proof for the $-$ case is same.

7.21 Standard form of Φ

We rewrite below the assignment Φ in a form in which Theorem 7.6 can be applied and use this to prove that Φ extends to an algebra homomorphism $U_\hbar(L\mathfrak{gl}_n) \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$.

Lemma *Let $y_j(v)$ be given by (7.3), and $\lambda_i^{\pm}(u)$ the operators of Proposition 7.5. Then,*

$$(\lambda_i^{\pm}(u) - 1)B(y_j(v)) = \pm(\delta_{i,j} - \delta_{j,i+1})\frac{e^{\hbar v}-1}{v}e^{uv}$$

The proof of this lemma essentially follows from Proposition 7.5.

Corollary *For any $i, i' \in \mathbf{I}$, we have*

$$\frac{\lambda_i^+(u)(\mathrm{Td}_{i'}^+(v))}{\mathrm{Td}_{i'}^+(u)} = \frac{\mathrm{Td}_{i'}^+(v)}{\lambda_i^-(u)(\mathrm{Td}_{i'}^+(v))} = \left(\frac{v-u+\hbar}{1-e^{-v+u-\hbar}}\frac{1-e^{-v+u}}{v-u}\right)^{\delta_{i,i'}-\delta_{i,i'-1}}$$

$$\frac{\lambda_i^+(u)(\mathrm{Td}_{i'}^-(v))}{\mathrm{Td}_{i'}^-(u)} = \frac{\mathrm{Td}_{i'}^-(v)}{\lambda_i^-(u)(\mathrm{Td}_{i'}^-(v))} = \left(\frac{v-u}{1-e^{-v+u}}\frac{1-e^{-v+u+\hbar}}{v-u-\hbar}\right)^{\delta_{i,i'}-\delta_{i,i'+1}}$$

It follows that

$$\Phi(E_{i,k}) = e^{k\sigma_i^+} g_i^+(\sigma_i^+)e_{i,0}$$
$$\Phi(F_{i,k}) = e^{k\sigma_i^-} g_i^-(\sigma_i^-)f_{i,0}$$

where

$$g_i^+(u) = q^{-\Delta_0-\theta_{i,0}}\frac{\hbar}{q-q^{-1}}\mathrm{Td}_i^+(u)$$
$$g_i^-(u) = q^{\Delta_0+\theta_{i+1,0}}\frac{\hbar}{q-q^{-1}}\mathrm{Td}_i^-(u) \tag{7.7}$$

7.22

We record the action of the operators $\lambda^{\pm}_{i'}(u)$ on $g^{\pm}_i(v)$ using Corollary 7.21

$$\lambda^+_i(u)(g^+_i(v)) = \lambda^-_{i-1}(u)(g^+_i(v)) = g^+_i(v)\frac{v-u+\hbar}{e^{v+\hbar/2}-e^{u-\hbar/2}}\frac{e^v-e^u}{v-u} \tag{7.8}$$

$$\lambda^+_{i+1}(u)(g^-_i(v)) = \lambda^-_i(u)(g^-_i(v)) = g^-_i(v)\frac{v-u-\hbar}{e^{v-\hbar/2}-e^{u+\hbar/2}}\frac{e^v-e^u}{v-u} \tag{7.9}$$

Using the fact that $\lambda^+_i(u)\lambda^-_i(u) = \mathrm{id}$, we get four more equations from these. Moreover, $\lambda^{\pm}_{i'}(u)(g^+_i(v)) = g^+_i(v)$ for $i' \neq i, i-1$ and $\lambda^{\pm}_{i'}(u)(g^-_i(v)) = g^-_i(v)$ for $i' \neq i, i+1$.

7.23

Theorem *The series $g^{\pm}_i(u)$ satisfy the conditions (A),(B),(C0)–(C2) of Theorem 7.6 and therefore give rise to an algebra homomorphism $\Phi : U_\hbar(L\mathfrak{gl}_n) \to \widehat{Y_\hbar(\mathfrak{gl}_n)}$.*

7.24 Proof of (A)

We need to prove that for every $i, i' \in \mathbf{I}$, we have

$$g^+_i(u)\lambda^+_i(u)(g^-_{i'}(v)) = g^-_{i'}(v)\lambda^-_{i'}(v)(g^+_i(u))$$

If $i \neq i', i'+1$, both sides are equal to $g^+_i(u)g^-_{i'}(v)$. For $i = i'$, by (7.8)–(7.9), the left- and right-hand sides are respectively equal to

$$\frac{v-u}{e^v-e^u}\frac{e^{v-\hbar/2}-e^{u+\hbar/2}}{v-u-\hbar} \quad \text{and} \quad \frac{u-v}{e^u-e^v}\frac{e^{u+\hbar/2}-e^{v-\hbar/2}}{u-v+\hbar}$$

The case $i = i'+1$ follows in the same way.

7.25 Proof of (B)

Let $i \in \mathbf{I}$. By (7.9),

$$\begin{aligned} g^+_i(u)\lambda^+_i(u)(g^-_i(u)) &= g^+_i(u)g^-_i(u)\frac{q-q^{-1}}{\hbar} \\ &= q^{\theta_{i+1,0}-\theta_{i,0}}\frac{\hbar}{q-q^{-1}}\mathrm{Td}^+_i(u)\mathrm{Td}^-_i(u) \\ &= \frac{\hbar}{q-q^{-1}}\exp\left(\frac{\hbar(\theta_{i+1,0}-\theta_{i,0})}{2}+td^+_i(u)+td^-_i(u)\right) \end{aligned}$$

By definition of $td_i^{\pm}$,

$$\begin{aligned} td_i^+(u) + td_i^-(u) &= -By_{i+1}(-\partial)J'(u) + By_i(-\partial)J'(u+\hbar) \\ &= -B(y_{i+1}(v) - y_i(v+\hbar))|_{v=-\partial}\, J'(u) \end{aligned}$$

where the second equality follows from $J'(u+\hbar) = \mathrm{e}^{\hbar\partial}J'(u)$ and the fact that $\mathrm{e}^{pv}B(f(u)) = B(f(u+p))$. Next, the definition of y_i yields

$$y_{i+1}(v) - y_i(v+\hbar) = d_{i+1}(v) - d_i(v)$$

hence

$$td_i^+(u) + td_i^-(u) = \hbar\sum_{r\geq 0}(-1)^{r+1}\frac{d_{i+1,r} - d_{i,r}}{r!}J^{(r+1)}(u)$$

which implies that

$$\frac{\hbar(\theta_{i+1,0} - \theta_{i,0})}{2} + td_i^+(u) + td_i^-(u) = \hbar\sum_{r\geq 0}(-1)^{r+1}\frac{d_{i+1,r} - d_{i,r}}{r!}G^{(r+1)}(u)$$

and the proof of (B) follows from Proposition 4.2

7.26 Proof of (C0)–(C2)

The condition (C0) follows from the fact that $(\lambda_i^{\pm}(u) - 1)(g_{i'}^{\pm}(v)) = 0$ if $|i - i'| > 1$. Since the proof of (C1) is the same as the one given in the verification of (A), we are left with checking (C2). We need to show that, for any $i \in \mathbf{I} \setminus \{n-1\}$,

$$g_i^+(u)\lambda_i^+(u)g_{i+1}^+(v)\frac{\mathrm{e}^u - \mathrm{e}^v}{u - v} = g_{i+1}^+(v)\lambda_{i+1}^{\pm}(v)g_i^+(u)\frac{\mathrm{e}^{u-\hbar/2} - \mathrm{e}^{v+\hbar/2}}{u - v - \hbar}$$

Using (7.8)–(7.9), the left- and right-hand sides are respectively equal to

$$\frac{u-v}{\mathrm{e}^u - \mathrm{e}^v}\,\frac{\mathrm{e}^{u-\hbar/2} - \mathrm{e}^{v+\hbar/2}}{u - v - \hbar}\,\frac{\mathrm{e}^u - \mathrm{e}^v}{u - v} \quad \text{and} \quad \frac{\mathrm{e}^{u-\hbar/2} - \mathrm{e}^{v+\hbar/2}}{u - v - \hbar}$$

7.27 Isomorphism between completions

We give below an analogue of Theorem 6.2 for $\mathfrak{g} = \mathfrak{gl}_n$. We begin by defining the appropriate completion of $U_\hbar(L\mathfrak{gl}_n)$, which differs from the one used in Sect. 6.3 due to the fact that $\mathfrak{gl}_n$ is not semisimple.

For each $r \geq 0$, $t \in \mathbb{Z}$ and $X = E_i, F_i, \Theta_j$, where $i \in \mathbf{I}$ and $j \in \mathbf{J}$, consider the element

$$X_{r;t} = \sum_{s=0}^{r} (-1)^s \binom{r}{s} X_{s+t}$$

where $\Theta_{j,l} = (\Theta^+_{j,l} - \Theta^-_{j,l})/(q - q^{-1})$. Note that $X_{r;t} = x \otimes z^t(1-z)^r \mod \hbar$ where $x \in \mathfrak{gl}_n$ is such that $X = x \mod \hbar$. Let $\mathcal{K}_r$ be the two-sided ideal of $U_\hbar(L\mathfrak{gl}_n)$ generated by the elements $\{X_{r';t}\}_{r' \geq r, t \in \mathbb{Z}}$, and $\hbar$ if $r = 1$. Finally, let $\mathcal{J}_n \subset U_\hbar(L\mathfrak{gl}_n)$ be the ideal

$$\mathcal{J}_m = \sum_{\substack{m_1, \ldots, m_k \geq 1 \\ m_1 + \cdots + m_k = m}} \mathcal{K}_{m_1} \ldots \mathcal{K}_{m_k}$$

Then, $\mathcal{J}_m$ is a descending filtration, $\mathcal{J}_m \mathcal{J}_{m'} \subset \mathcal{J}_{m+m'}$, and the completion

$$\widehat{U_\hbar(L\mathfrak{gl}_n)} = \varprojlim U_\hbar(L\mathfrak{gl}_n)/\mathcal{J}_m$$

is a flat deformation of the completion of $U(\mathfrak{gl}_n[z, z^{-1}])$ at $z = 1$.

Remark The reason we use $\mathcal{J}_m$ instead of the powers of the evaluation ideal $\mathcal{J}_1$ as in Sect. 6.3 can be seen at the classical level. Indeed, the correct filtration of $U(\mathfrak{gl}_n[z, z^{-1}])$ is given by the $J_m = U((z-1)^m \mathfrak{gl}_n[z, z^{-1}])$ which contains, but does not equal J_1^m. For example, if I is the identity matrix, then $I \otimes (z-1)^2 \in J_2 \setminus \bigcup_{m \geq 1} J_1^m$.

For each $m \in \mathbb{N}$, set

$$\widehat{Y_\hbar(\mathfrak{gl}_n)}_{\geq m} = \prod_{m' \geq m} (Y_\hbar(\mathfrak{gl}_n))_{m'} \subset \widehat{Y_\hbar(\mathfrak{gl}_n)}$$

where $(Y_\hbar(\mathfrak{gl}_n))_m$ is the subspace of degree m. The proof of the following result is similar to that of Theorem 6.2 and therefore omitted.

Theorem *The homomorphism Φ maps $\mathcal{J}_m$ to $\widehat{Y_\hbar(\mathfrak{gl}_n)}_m$ for any $m \in \mathbb{N}$ and induces an isomorphism of completed algebras*

$$\widehat{\Phi} : \widehat{U_\hbar(L\mathfrak{gl}_n)} \xrightarrow{\sim} \widehat{Y_\hbar(\mathfrak{gl}_n)}$$

8 Appendix: Proof of the Serre relations

8.1

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra. The aim of this appendix is to prove the following

Proposition *Let Φ be the assignment $\{E_{i,k}, F_{i,k}, H_{i,k}\} \to \widehat{Y_\hbar(\mathfrak{g})}$ given in Sects. 3.1–3.3 and assume that the relations (A) and (B) of Theorem 3.4 hold. Then, Φ is compatible with the q-Serre relations (QL6).*

For $i \neq j \in \mathbf{I}$, set $m = 1 - a_{ij}$. Define, for any $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ and $l \in \mathbb{Z}$

$$\mathcal{S}^q_{ij}(\underline{k}, l) = \sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \Phi(E_{i,k_{\pi(1)}}) \cdots \Phi(E_{i,k_{\pi(m-s)}}) \Phi(E_{j,l}) \Phi(E_{i,k_{\pi(m-s+1)}}) \cdots \Phi(E_{i,k_{\pi(m)}}) \in \widehat{Y_\hbar(\mathfrak{g})} \tag{8.1}$$

and let $\mathcal{S}^q_{ij} = \mathcal{S}^q_{ij}(\underline{0}, 0)$, explicitly given as follows

$$\mathcal{S}^q_{ij} = \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \left(\Phi(E_{i,0})\right)^{m-s} \Phi(E_{j,0}) \left(\Phi(E_{i,0})\right)^s \tag{8.2}$$

Our aim is to show that $\mathcal{S}^q_{ij}(\underline{k}, l) = 0$. Let us outline the main steps of the proof.

(1) We first reduce the proof of $\mathcal{S}^q_{ij}(\underline{k}, l) = 0$ to $\mathcal{S}^q_{ij} = 0$. This is achieved in Lemma 8.5.
(2) By a standard argument using the representation theory of $U_\hbar\mathfrak{sl}_2$, we deduce in Lemma 8.6 that $\mathcal{S}^q_{ij}$ acts by zero on any finite-dimensional representation of $Y_\hbar(\mathfrak{g})$.
(3) Finally, we show that these representations separate points in $Y_\hbar(\mathfrak{g})$, and hence that $\mathcal{S}^q_{ij} = 0$. Sections 8.8 and 8.9 are devoted to the proof of this fact (Corollary 8.9) which was communicated to us by V. G. Drinfeld.

8.2 The algebra $\overline{Y}$

Define an auxiliary algebra $\overline{Y}$ to be the unital, associative $\mathbb{C}[\hbar]$-algebra generated by $\{\overline{\xi}_{i,r}, \overline{x}_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$ subject to the following relations

(1) For every $i, j \in \mathbf{I}$ and $r, s \in \mathbb{N}$

$$[\overline{\xi}_{i,r}, \overline{\xi}_{j,s}] = 0$$

(2) For every $i, j \in \mathbf{I}$ and $s \in \mathbb{N}$

$$[\overline{\xi}_{i,0}, \overline{x}_{j,s}] = d_i a_{ij} \overline{x}_{j,s}$$

(3) For every $i, j \in \mathbf{I}$ and $r, s \in \mathbf{I}$

$$[\overline{\xi}_{i,r+1}, \overline{x}_{j,s}] - [\overline{\xi}_{i,r}, \overline{x}_{j,s+1}] = \frac{d_i a_{ij} \hbar}{2} (\overline{\xi}_{i,r} \overline{x}_{j,s} + \overline{x}_{j,s} \overline{\xi}_{i,r})$$

We denote by $\overline{Y}^0 \subset \overline{Y}$ the commutative subalgebra generated by $\{\overline{\xi}_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$ and by $\overline{Y}^{>0}$ the subalgebra of $\overline{Y}$ generated by $\{\overline{x}_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{N}}$. The latter is a free $\mathbb{C}[\hbar]$-algebra over this set of generators. Moreover, by Corollary 2.5, $\overline{Y} \cong \overline{Y}^0 \otimes \overline{Y}^{>0}$.

8.3 The operators $\overline{\sigma}_{i,(k)}$ and $\overline{\sigma}_j$

The algebra $\overline{Y}$ has a grading by the root lattice Q given by

$$\deg(\overline{\xi}_{i,r}) = 0 \qquad \text{and} \qquad \deg(\overline{x}_{i,r}) = \alpha_i$$

Fix henceforth $i \neq j \in \mathbf{I}$, set $m = 1 - a_{ij}$ and let $\overline{Y}_{m\alpha_i+\alpha_j}$ be the homogeneous component of $\overline{Y}$ of degree $m\alpha_i + \alpha_j$.

Define operators $\overline{\sigma}_j, \overline{\sigma}_{i,(k)}$ on $\overline{Y}_{m\alpha_i+\alpha_j}$ as follows. Since $\overline{Y}_{m\alpha_i+\alpha_j} \cong \overline{Y}^0 \otimes \overline{Y}^{>0}_{m\alpha_i+\alpha_j}$ and $\overline{Y}^{>0}$ is free, we have

$$\overline{Y}^{>0}_{m\alpha_i+\alpha_j} \cong \overline{Y}^0 \otimes \bigoplus_{s=0}^{m} \overline{Y}(i)^{\otimes m-s} \otimes \overline{Y}(j) \otimes \overline{Y}(i)^{\otimes s}$$

where, for $a = i, j$, $\overline{Y}(a) = \overline{Y}^{>0}_{\alpha_a}$ is spanned by $\{\overline{x}_{a,r}\}_{r\in\mathbb{N}}$. Let $\overline{\sigma}_a$ denote the $\mathbb{C}[\hbar]$-linear map on $\overline{Y}(a)$ given by $\overline{\sigma}_a(x_{a,r}) = x_{a,r+1}$. For any $k = 1, \ldots, m$, define the Y^0-linear operator $\overline{\sigma}_{i,(k)}$ on $\overline{Y}_{m\alpha_i+\alpha_j}$ by letting it act on the summand $\overline{Y}(i)^{\otimes m-s} \otimes \overline{Y}(j) \otimes \overline{Y}^{\otimes s}(i)$ as

$$\overline{\sigma}_{i,(k)} = \begin{cases} 1^{\otimes k-1} \otimes \overline{\sigma}_i \otimes 1^{\otimes m+1-k} & \text{if } k \leq m-s \\ 1^{\otimes k} \otimes \overline{\sigma}_i \otimes 1^{\otimes m-k} & \text{otherwise} \end{cases}$$

Similarly, let $\sigma_j \in \mathrm{End}_{\overline{Y}^0}(\overline{Y}_{m\alpha_i+\alpha_j})$ be given by $1^{\otimes m-s} \otimes \overline{\sigma}_j \otimes 1^{\otimes s}$ on $\overline{Y}(i)^{\otimes m-s} \otimes \overline{Y}(j) \otimes \overline{Y}(i)^{\otimes s}$.

8.4 The projection $\mathbf{p_{ij}}$

Let $p : \overline{Y} \to Y_\hbar(\mathfrak{g})$ be the algebra homomorphism obtained by sending $\overline{\xi}_{a,r} \longmapsto \xi_{a,r}$ and $\overline{x}_{a,r} \longmapsto x^+_{a,r}$ for every $a \in \mathbf{I}$ and $r \in \mathbb{N}$, and let p_{ij} be the restriction of p to $\overline{Y}_{m\alpha_i+\alpha_j}$. The following holds by Proposition 2.8.

Lemma *The kernel of p_{ij} is the $\mathbb{C}[\hbar]$-linear span of the following elements*

(1) *For any* $0 \leq s \leq m-1$ *and* $A(u_1, \ldots, u_m, w) \in \overline{Y}^0[u_1, \ldots, u_m, w]$

$$A(\overline{\sigma}_{i,(1)}, \ldots, \overline{\sigma}_{i,(m)}, \overline{\sigma}_j)\Big((\overline{\sigma}_{i,(m-s)} - \overline{\sigma}_j - a\hbar)\overline{x}_{i,0}^{m-s}\overline{x}_{j,0}\overline{x}_{i,0}^{s} \\ -(\overline{\sigma}_{i,(m-s)} - \overline{\sigma}_j + a\hbar)\overline{x}_{i,0}^{m-s-1}\overline{x}_{j,0}\overline{x}_{i,0}^{s+1}\Big)$$

where $a = d_i a_{ij}/2$.

(2) *For any* $0 \leq s \leq m$, $k \in \{1, \dots, m-1\} \setminus \{m-s\}$ *and* $A(u_1, \dots, u_m, w) \in \overline{Y}^0[u_1, \dots, u_m, w]^{(k\,k+1)}$

$$A(\overline{\sigma}_{i,(1)}, \dots, \overline{\sigma}_{i,(m)}, \overline{\sigma}_j)(\overline{\sigma}_{i,(k)} - \overline{\sigma}_{i,(k+1)} - d_i\hbar)\overline{x}_{i,0}^{m-s}\overline{x}_{j,0}\overline{x}_{i,0}^{s}$$

(3) *For every* $A(u_1, \dots, u_m, w) \in \overline{Y}^0[u_1, \dots, u_m, w]^{\mathfrak{S}_m}$

$$A(\overline{\sigma}_{i,(1)}, \dots, \overline{\sigma}_{i,(m)}, \overline{\sigma}_j)\left(\sum_{s=0}^{m}(-1)^s \binom{m}{s} \overline{x}_{i,0}^{m-s}\overline{x}_{j,0}\overline{x}_{i,0}^{s}\right)$$

Corollary *Let* $X \in Ker(p_{ij})$ *and* $A(u_1, \dots, u_m, w) \in \overline{Y}^0[u_1, \dots, u_m, w]^{\mathfrak{S}_m}$. *Then,*

$$A(\overline{\sigma}_{i,(1)}, \dots, \overline{\sigma}_{i,(m)}, \overline{\sigma}_j)X \in Ker(p_{ij})$$

8.5 Reduction step

Let $\overline{\mathcal{S}^q_{ij}}(\underline{k}, l)$, $\overline{\mathcal{S}^q_{ij}}$ denote the elements of $\overline{Y}_{m\alpha_i+\alpha_j}$ defined by the same expressions as (8.1)–(8.2). Then,

$$\overline{\mathcal{S}^q_{ij}}(\underline{k}, l) = \left(\sum_{\pi\in\mathfrak{S}_m} \mathrm{e}^{k_{\pi(1)}\overline{\sigma}_{i,(1)}} \dots \mathrm{e}^{k_{\pi(m)}\overline{\sigma}_{i,(m)}}\mathrm{e}^{l\overline{\sigma}_j}\right)\overline{\mathcal{S}^q_{ij}}$$

Using Corollary 8.4, we obtain the following

Lemma $\mathcal{S}^q_{ij} = 0$ *implies* $\mathcal{S}^q_{ij}(\underline{k}, l) = 0$ *for every* $k_1, \dots, k_m, l \in \mathbb{Z}$.

8.6

By a finite-dimensional representation of $\widehat{Y_\hbar(\mathfrak{g})}$, we shall mean a finitely-generated topologically free $\mathbb{C}[[\hbar]]$-module endowed with a $\mathbb{C}[[\hbar]]$-linear action of $\widehat{Y_\hbar(\mathfrak{g})}$.

Lemma *Let* $\mathcal{V}$ *be a finite-dimensional representation of* $\widehat{Y_\hbar(\mathfrak{g})}$. *Then,* $\mathcal{S}^q_{ij}$ *acts by zero on* $\mathcal{V}$.

Proof Let $\mathcal{U}_i$ be the subalgebra of $\widehat{Y_\hbar(\mathfrak{g})}$ generated by

$$\mathcal{E}_i = \Phi(E_{i,0}) \qquad \mathcal{F}_i = \Phi(F_{i,0}) \qquad \mathcal{H}_i = \Phi(H_{i,0})$$

By Lemma 3.5, $\{\mathcal{E}_i, \mathcal{F}_i, \mathcal{H}_i\}$ satisfy the defining relations of the quantum group $U_{\hbar_i}\mathfrak{sl}_2$, where $\hbar_i = d_i\hbar/2$. We use the following notation of q-adjoint operator (see [19,

§4.18]) which gives a representation of $\mathcal{U}_i$ on any algebra containing it

$$\begin{aligned}
\mathrm{ad}_q(\mathcal{E}_i)(X) &= \mathcal{E}_i X - \mathcal{K}_i X \mathcal{K}_i^{-1} \mathcal{E}_i \\
\mathrm{ad}_q(\mathcal{F}_i)(X) &= \mathcal{F}_i X \mathcal{K}_i - X \mathcal{F}_i \mathcal{K}_i \\
\mathrm{ad}_q(\mathcal{H}_i)(X) &= [\mathcal{H}_i, X]
\end{aligned}$$

where $\mathcal{K}_i = q_i^{\mathcal{H}_i} = \mathrm{e}^{\hbar_i \mathcal{H}_i}$. Let $\rho : \widehat{Y_\hbar(\mathfrak{g})} \to \mathrm{End}(\mathcal{V})$ be the representation. Then,

$$\begin{aligned}
\mathrm{ad}_q(\rho(\mathcal{F}_i))\rho(\mathcal{E}_j) &= 0 \\
\mathrm{ad}_q(\rho(\mathcal{H}_i))\rho(\mathcal{E}_j) &= a_{ij}\rho(\mathcal{E}_j)
\end{aligned}$$

where the first identity follows from Lemma 3.5. Thus, as a $\mathcal{U}_i$-module, $\mathrm{End}(\mathcal{V})$ contains the lowest weight vector $\rho(\mathcal{E}_j)$ of weight a_{ij}. By the representation theory of $U_{\hbar_i}\mathfrak{sl}_2$, we get

$$\mathrm{ad}_q(\rho(\mathcal{E}_i))^m \rho(\mathcal{E}_j) = \rho\left(\mathrm{ad}_q(\mathcal{E}_i)^m \mathcal{E}_j\right) = 0$$

and the assertion follows from the well-known identity (see [19, Lemma 4.18])

$$\mathrm{ad}_q(\mathcal{E}_i)^m \mathcal{E}_j = \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} \mathcal{E}_i^{m-s} \mathcal{E}_j \mathcal{E}_i^s$$

□

8.7

Let $I_\hbar \subset Y_\hbar(\mathfrak{g})$ be the ideal defined by

$$I_\hbar = \bigcap_{(\mathcal{V},\rho)} \mathrm{Ker}(\rho)$$

where $\mathcal{V}$ runs over all finite-dimensional *graded* modules over $Y_\hbar(\mathfrak{g})$, that is finitely-generated torsion-free $\mathbb{C}[\hbar]$-modules admitting a $\mathbb{C}[\hbar]$-linear action $\rho : Y_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathcal{V})$ and a $\mathbb{Z}$-grading compatible with that on $Y_\hbar(\mathfrak{g})$.

Lemma $I_\hbar = 0$ *implies* $\mathcal{S}^q_{ij} = 0$.

Proof The action of $Y_\hbar(\mathfrak{g})$ on any finite-dimensional graded module $\mathcal{V}$ extends to one of $\widehat{Y_\hbar(\mathfrak{g})}$ on the completion $\widehat{\mathcal{V}}$ of $\mathcal{V}$ with respect to its grading. By Lemma 8.6, $\mathcal{S}^q_{ij}$ acts by 0 on $\widehat{\mathcal{V}}$ and therefore so do its homogeneous components $\mathcal{S}^q_{ij;n} \in Y_\hbar(\mathfrak{g}),\ n \geq 0$ on $\mathcal{V}$. Thus, $\mathcal{S}^q_{ij;n} \in I_\hbar$ for any n and $\mathcal{S}^q_{ij} = 0$. □

8.8

The following result, and its proof are due to Drinfeld [12]

Proposition *The ideal* $I_\hbar \subset Y_\hbar(\mathfrak{g})$ *is trivial.*

Proof It suffices to show that $I = I_\hbar/\hbar I_\hbar$ is trivial. Indeed, if $I_\hbar = \hbar I_\hbar$, then $I_\hbar = \bigcap_k \hbar^k I_\hbar \subset \bigcap_k \hbar^k Y_\hbar(\mathfrak{g}) = 0$. By definition of $I_\hbar$, $I_\hbar \cap \hbar Y_\hbar(\mathfrak{g}) = \hbar I_\hbar$ so that I embeds into $Y_\hbar(\mathfrak{g})/\hbar Y_\hbar(\mathfrak{g}) = U(\mathfrak{g}[s])$. Since graded representations are stable under tensor product, $I_\hbar$ is a Hopf ideal of $Y_\hbar(\mathfrak{g})$, that is

$$\Delta(I_\hbar) \subset Y_\hbar(\mathfrak{g}) \otimes I_\hbar + I_\hbar \otimes Y_\hbar(\mathfrak{g})$$

It follows that I is a co-Poisson Hopf ideal of $U(\mathfrak{g}[s])$. By Corollary 8.9 below, any such ideal is either trivial or equal to $U(\mathfrak{g}[s])$. Since $Y_\hbar(\mathfrak{g})$ possesses non-trivial finite-dimensional graded representations, for example the action on $\mathbb{C}[\hbar]$ given by the counit, $I_\hbar$ is a proper ideal of $Y_\hbar(\mathfrak{g})$ and is therefore equal to zero. □

8.9

Recall that a co-Poisson Hopf algebra A is a Hopf algebra together with a Poisson cobracket $\delta : A \to A \wedge A$ satisfying the following compatibility condition (see [5, §6.2] for details):

$$\delta(xy) = \delta(x)\Delta(y) + \Delta(x)\delta(y)$$

For a Lie algebra $\mathfrak{a}$, there is a one-to-one correspondence between co-Poisson structures on $U\mathfrak{a}$ and Lie bialgebra structures on $\mathfrak{a}$ [5, Proposition 6.2.3]. Moreover, there is a one-to-one correspondence between co-Poisson Hopf ideals of $U\mathfrak{a}$ and Lie bialgebra ideals of $\mathfrak{a}$.

The Lie bialgebra structure on $\mathfrak{g}[s]$ is given by

$$\begin{aligned}\delta &: \mathfrak{g}[s] \to \mathfrak{g}[s] \otimes \mathfrak{g}[s] \cong (\mathfrak{g} \otimes \mathfrak{g})[s,t]\\ \delta(f)(s,t) &= (\mathrm{ad}(f(s)) \otimes 1 + 1 \otimes \mathrm{ad}(f(t)))\left(\frac{\Omega}{s-t}\right)\end{aligned} \tag{8.3}$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir tensor. Note that δ lowers the degree by 1.

Let $\mathfrak{a} \subset \mathfrak{g}[s]$ be the Lie bialgebra ideal corresponding to co-Poisson Hopf ideal $I \subset U(\mathfrak{g}[s])$. By the discussion given in previous paragraph,

$$\delta(\mathfrak{a}) \subset \mathfrak{a} \otimes \mathfrak{g}[s] + \mathfrak{g}[s] \otimes \mathfrak{a} \tag{8.4}$$

Lemma *Let* $\mathfrak{a} \subset \mathfrak{g}[s]$ *be an ideal. Then* $\mathfrak{a}$ *is of the form* $\mathfrak{a} = \mathfrak{g} \otimes g\mathbb{C}[s]$ *for some polynomial* $g \in \mathbb{C}[s]$.

Proof Let $S \subset \mathbb{C}[s]$ be the set of all polynomials f such that there exists some non-zero $x \in \mathfrak{g}$ for which $x \otimes f \in \mathfrak{a}$. We claim that S is an ideal of $\mathbb{C}[s]$. Let $f \in S$ and $g \in \mathbb{C}[s]$. Let $0 \neq x \in \mathfrak{g}$ be such that $x \otimes f \in \mathfrak{a}$, and choose $y \in \mathfrak{g}$ such that $[x, y] \neq 0$. Then

$$[x, y] \otimes fg = [x \otimes f, y \otimes g] \in \mathfrak{a}$$

and hence $fg \in S$.

Now for any $f \in S$, the set $\{x \in \mathfrak{g} : x \otimes f \in \mathfrak{a}\}$ is an ideal in $\mathfrak{g}$, which is non-zero and hence equal to $\mathfrak{g}$. This proves that $\mathfrak{a} = \mathfrak{g} \otimes S$. Since $\mathbb{C}[s]$ is a principal ideal domain, the lemma is proved. □

Corollary *Let* $\mathfrak{a} \subset \mathfrak{g}[s]$ *be a Lie bialgebra ideal. Then either* $\mathfrak{a} = 0$ *or* $\mathfrak{a} = \mathfrak{g}[s]$.

Proof Let $g \in \mathbb{C}[s]$ be such that $\mathfrak{a} = \mathfrak{g} \otimes (g) \subset \mathfrak{g}[s]$. By (8.3), we know that the Lie cobracket δ lowers the degree by 1. Using (8.4), we conclude that g is a constant polynomial. □

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